Broad relaxation spectrum and the field theory of glassy dynamics for pinned elastic systems

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We study thermally activated, low-temperature equilibrium dynamics of elastic systems pinned by disorder using one loop functional renormalization group (FRG). Through a series of increasingly complete approximations, we investigate how the field theory reveals the glassy nature of the dynamics, in particular divergent barriers and barrier distributions controling the spectrum of relaxation times. First, we naively assume a single relaxation time τ_k for each wave vector k, leading to analytical expressions for equilibrium dynamical response and correlations. These exhibit two distinct scaling regimes (scaling variables $Tk^{\theta}\ln t$ and t/τ_k, respectively, with T the temperature, θ the energy fluctuation exponent, and $\tau_k \sim e^{ck^{-\theta}/T}$) and are easily extended to quasi-equilibrium and aging regimes. A careful study of the dynamical operators encoding for fluctuations of the relaxation times shows that this first approach is unsatisfactory. A second stage of approximation including these fluctuations, based on a truncation of the dynamical effective action to a random friction model, yields a size (L) dependent log-normal distribution of relaxation times (effective barriers centered around L^{θ} and of fluctuations $\sim L^{\theta/2}$) and some procedure to estimate dynamical scaling functions. Finally, we study the full structure of the running dynamical effective action within the field theory. We find that relaxation time distributions are nontrivial (broad but not log normal) and encoded in a closed hierarchy of FRG equations divided into levels $p=0,1,\ldots$, corresponding to vertices proportional to the pth power of frequency ω^p. We show how each level p can be solved independently of higher ones, the lowest one ($p=0$) comprising the statics. A thermal boundary layer ansatz (TBLA) appears as a consistent solution. It extends the one discovered in the statics which was shown to embody droplet thermal fluctuations. Although perturbative control remains a challenge, the structure of the dynamical TBLA which encodes barrier distributions opens the way for deeper understanding of the field theory approach to glasses.

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I. INTRODUCTION

Extremely slow dynamics is a ubiquitous property of complex and disordered materials. Despite many decades of research, current understanding of such *glassy* motion is limited to phenomenological models [1], mean-field theory [2–4], and abstract caricatures in terms of the dynamics of small numbers of degrees of freedom in a complex energy landscape [5,6]. In addition, exact solutions in one dimension (e.g., for the random field Ising model [7]) little is known about the nonequilibrium behavior of realistic models. True disordered materials, from spin glasses to supercooled liquids to the pinned elastic medium, involve extensive numbers of local degrees of freedom such as atoms and spins moving *collectively* in a random environment [8–10], with either external or self-induced randomness. The pinned elastic medium being the simplest model involving such physics we study it as a prototype. It is of interest by itself for numerous experimental systems such as vortex lattices in superconductors [10,11], interfaces in magnets [12,13], charge density waves [14], and Wigner crystals [15]. The equation of motion is

$$\eta\partial_t u_{rt} = c\nabla_r^2 u_{rt} + f(u_{rt},r) + \zeta(r,t), \tag{1}$$

where $u(r)$ is a height (or displacement) field, η a bare frictional damping coefficient, c is the elastic modulus, $f(u,r)$ is the quenched random pinning force, and $\zeta(r,t)$ is a thermal noise. Here r is he d-dimensional internal coordinate of the elastic object. Both f and ζ are Gaussian random variables with zero mean and second moment

$$\overline{f(u,r)f(u',r')} = \Delta(u-u')\delta^d(r-r'), \tag{2}$$

$$\langle\zeta(r,t)\zeta(r',t')\rangle = 2\eta T\delta^d(r-r')\delta(t-t'), \tag{3}$$

where T is the temperature and we set Boltzmann's constant $k_B=1$. The value of η generally sets the relaxation time scale, e.g., here $\eta=t_0 c\Lambda^2$, t_0 being the microscopic time scale, and Λ the short scale momentum cutoff. In general, one may be interested in a variety of thermal and sample-to-sample fluctuations of the system, as well as various responses of the system to external probes. We will focus on the simplest of the latter, described by (linear) response functions

$$R_{rr'tt'} = \left\langle \frac{\partial u_r(t)}{\partial\zeta_{r'}(t')} \right\rangle, \tag{4}$$

in a given disorder realization, and its disorder average

$$\overline{\mathcal{R}_{rr'tt'}} = R_{r-r',tt'}. \tag{5}$$

At equilibrium, both the single sample and the averaged response functions become time translationally invariant, $\mathcal{R}_{r-r',tt'} = \mathcal{R}_{r-r',t-t'}$ and $R_{r-r',tt'} = R_{r-r',t-t'}$.

Equation (1) has the usual Langevin form, and guarantees the existence of a stable equilibrium probability distribution, provided, as assumed here, that $f(u,r) = -\partial V(u,r)/\partial u$ is of gradient form. The equilibrium distribution (strictly speaking defined in a finite size sample) has the Boltzmann form $p(u) \propto \exp(-H[u]/T)$, with

$$H[u] = \int d^d\mathbf{r}\left[\frac{c}{2}|\nabla u|^2 + V(u(\mathbf{r}),\mathbf{r})\right]. \tag{6}$$

Three universality classes of special interest are usually considered: (i) short range disorder $\Delta(u)$ which describes, e.g., random bond (RB) disorder for magnetic domain walls, (ii) long range disorder which describes, e.g., random field (RF) disorder, and (ii) random periodic (RP) $\Delta(u)$ which describes pinned density waves or lattices. Although these systems differ in their details, e.g., in their roughness exponent $u \sim r^\zeta$, they do not yield qualitatively different behavior in their dynamical response studied here.

The aim of the present paper is to develop an approach based on the renormalization group (RG) to study the low-temperature dynamics of pinned elastic systems described by Eq. (1). Although we focus on equilibrium dynamics, some of our considerations are also relevant for nonequilibrium relaxation. It was shown that to describe the statics at equilibrium one needs to follow the full correlator of the random potential (or the random force) using a functional RG (FRG) method in a $d=4-\epsilon$ expansion [16,17]. Several extensions describe correlated disorder [18], the driven dynamics near depinning [19–21] and the small applied force, thermally activated, creep regime [22]. However, until now the FRG has not been used to study the dynamical response and correlations in equilibrium or aging regimes, nor to probe the crucial question of the distribution of the relaxation times. These are important quantities directly probed in experiments where the system is often dominated by fluctuations or not able to reach equilibrium within the experimental time scales. We investigate this problem in three stages, of increasing accuracy (and, unfortunately, complexity), only in the last stage attempt is made to be exhaustive. A companion paper (Ref. [23]) is devoted to the statics. A short account of both works can be found in Ref. [24]. Some of the present considerations concerning approximate schemes have also been discussed independently in Ref. [25].

The first question investigated in the present paper is the validity of the single time scale approximation within the RG. Specifically, in the first part of our study (Sec. II) we use, as was done in previous works [19,20,22], the simplifying hypothesis that the relaxation of each internal mode, of wave vector k, is controlled by a single relaxation time scale τ_k. This allows us to obtain closed equations, within the one loop FRG, for the general two time response and correlations, as measured in aging experiments. It yields, in the equilibrium regime on which focus from then on, interesting analytical expressions for the equilibrium response and correlation which exhibit two distinct scaling regimes with scaling variables $k^\theta \ln t$ and t/τ_k, respectively ($\theta = d-2+2\zeta$ is the energy fluctuation exponent and $\tau_k \sim e^{ck^{-\theta}/T}$). However, several features of these results are found to be unsatisfactory, such as the nonmonotonicity of the response as a function of wave vector. A more complete description including time scale fluctuations thus appears necessary.

That sample to sample fluctuations should play an important role both in the statics and dynamics of disordered glasses is indeed expected from phenomenological arguments, e.g., the droplet scenario [1,26], which appears to describe simpler models such as Eq. (6) relatively well, at least in low dimensions [27]. Let us recall its main conclusions. In its simplest form, it supposes the existence, at each length scale L, of a small number of excitations of size $\delta\Phi \sim L^\zeta$ above a ground state, drawn from an energy distribution of width $\delta E \sim L^\theta$ with constant weight near $\delta E = 0$. While typically the elastic manifold is localized near a ground state, disorder averages of static thermal fluctuations at a given scale are dominated by rare samples/regions with two nearly degenerate minima. For example, as a simple but remarkable consequence, the $(2n)$th moment of u fluctuations is expected to behave as

$$\overline{(\langle u^2\rangle - \langle u\rangle^2)^n} \sim c_n(T/L^\theta)L^{2n\zeta}. \tag{7}$$

The droplet picture supposes that the long-time equilibrium dynamics is dominated by thermal activation between these quasidegenerate minima controlled by barriers of typical scale $U_L \sim L^\psi$. Little is known about the distribution of these barriers, but there is some evidence [28–30] that $\psi \approx \theta$. Even a modest distribution of barriers, however, due to the Arrhenius law $\tau_L \sim e^{U_L/T}$, yields relaxation time scales with an extremely broad distribution as $T \to 0$. Some probes of this broad distribution are the relaxation time moments which may be defined in a variety of ways. One begins by defining the relaxation time moments in a single sample,

$$\langle t^n\rangle_L = L^{-(d+2)}\int_0^\infty dt \int_{rr'}^L t^n \mathcal{R}_{rr't}, \tag{8}$$

which, for a particular disorder realization, describe the response of the center of mass coordinate to a (spatially) uniform force. For $n=1$ ($\langle t\rangle_L$) this gives one definition of the relaxation time of a single sample. A set of disorder-averaged moments may be obtained then by directly averaging the above objects $\overline{\langle t^n\rangle_L}$ giving the averaged response of the system

$$\overline{\langle t^n\rangle_L} = q^2 \left.\int_0^\infty dt t^n R_{q,t}\right|_{q=1/L}. \tag{9}$$

Alternatively, the *distribution* (from sample to sample) of the unaveraged relaxation time $\langle t\rangle_L$ is described by a second set of moments $\overline{\langle t\rangle_L^n}$. In general, one may construct many such objects scaling dimensionally as t^n but with different physical content. Mathematically, this is accomplished by averaging arbitrary products of the single sample moments, i.e., $\overline{\langle t^{p_1}\rangle_L \cdots \langle t^{p_N}\rangle_L}$, with $\Sigma_{j=1}^N p_j = n$. Any of these "nth" moments may behave as $\sim e^{\alpha(n)U_0 L^{\theta}/T}$ with $\alpha(n) \geqslant n$, or grow even faster with $\ln \overline{\tau^n} \gg L^{\theta}$, nor is it clear that the different definitions for a given n exhibit the same growth. Indeed, we will ultimately find different operators in a dynamical field theoretic formulation corresponding to each different type of moment, and some indications that indeed different growth rates obtain for each of these. A theory of these time scales is crucial to understanding both equilibrium response and correlations and to near-equilibrium phenomena such as creep [10,11,13,26]. Extensive calculations are possible in certain zero-dimensional toy models [49]. Although some analytical results have been obtained for $d \geqslant 1$ within mean-field limits [31–33], these do not include thermal activation over divergent barriers ($U_L \sim L^{\theta}$). The FRG [16,19–21], on the other hand, extended to nonzero temperature [18,22], seems to capture, already at the level of the single time scale approximation the existence of these growing barriers. However, until now neither the rare events nor fluctuating barriers have been obtained in this approach.

In the main part of our study (Secs. III and IV) we thus investigate how relaxation time distributions appear within the FRG. We first show that the equation of motion (1) generates under coarse graining a *random friction* term $\eta(r)$ (equivalently a random relaxation time $\tau \sim 1/\eta$). It is then natural to define, as a toy model, a random friction model which, in the absence of pinning disorder possesses a manifold of fixed points indexed by the full coarse grained probability distribution of the friction. A highly nontrivial question is how this distribution will flow under RG due to feedback from nonlinear terms when pinning disorder is reintroduced. We consider this question at the one loop level, in two stages.

In Sec. III we present a highly simplified analysis, but with the merit of explicitly exhibiting the broadening of the barrier distribution and allowing for some analytical expressions. It yields asymptotically a log-normal distribution of relaxation times, i.e., a nearly Gaussian distribution of effective barriers centered around L^{θ} and of typical fluctuations $\sim L^{\theta/2}$. Such a log-normal tail corresponds to the moment exponents $\alpha(n) = 2n^2 - n$. This is compared with numerical results [28–30] in the case of a directed polymer, where the width was fitted to $\sim L^{\theta}$. The question of the width is important since a width $\sim L^{\theta/2}$ is not expected to be large enough to modify the creep exponent, as can be seen from reexamining the calculations of Ref. [22] while a $\sim L^{\theta}$ width [29] would pose a problem to this order. We derive, within the same approximate scheme (Appendix E), closed equations for correlations and response functions (the fact that the width grows very fast can be exploited in a resummation of the fastest growing terms in the dynamical part of the Martin Siggia Rose functional). We find that the broadening is sufficiently fast to invalidate some of the previous analysis, e.g., the existence of a t/τ_k scaling regime.

Section IV contains the full systematic analysis of the running dynamical effective action. It is found that relaxation times distributions are determined by a closed hierarchy of FRG equations, each level p corresponding to an increasing power of frequency ω^p can be solved independently of higher ones, the lowest one being the statics $p=0$. This hierarchy involves functions parametrizing the local cross correlations between pinning disorder and random relaxation times. The previous approximation corresponds to projecting these FRG functions to their values at zero, while in fact the full set of nonlinear differential equations obeyed by these functions need to be solved, a formidably complex task. A thermal boundary layer ansatz (TBLA) appears to be a consistent solution. It extends the one discovered in the statics which was shown to reproduce droplet theory type behavior in thermal fluctuations. Here it yields a natural growth for moments of relaxation times measured by nontrivial exponents $\alpha(n) \neq 2n^2 - n$ determined by eigenvalue problems. Although perturbative control remains a challenge, the structure of the dynamical TBLA which encodes for barrier distributions opens the way for deeper understanding of the field theory approach to glasses.

The detailed outline of the paper is as follows. In Sec. II we recall the standard results of the FRG for the equilibrium dynamics using a single relaxation time approach. We then give a qualitative derivation of the two scaling regimes for the equilibrium response and correlation functions. The detailed equations obeyed by these functions are derived using a Wilson scheme in Appendix A and their analytical form is analyzed in the equilibrium regime in Appendix A and in the aging regime in Appendix B. In Sec. III we go beyond the single relaxation time approach. The random friction model is introduced in Sec. III B. We then incorporate pinning disorder in an approximate way in Sec. III C, analyze the resulting distribution of relaxation times and show that it become broad. The breakdown of the $\omega\tau_k$ scaling is analyzed in Sec. III D. In Sec. IV we discuss the systematics of the structure of the dynamical effective action. It does contain the statics which its recalled, together with its thermal boundary layer ansatz solution, in its equilibrium dynamics formulation in Sec. IV A. Then in Sec. IV B–IV E we display the hierarchical structure of the FRG equations and how a generalized thermal boundary later structure appears as a consistent solution determining the growth of the moments of the relaxation times through nontrivial eigenvalue problems. We conclude in Sec. V with some general remarks. Finally, a set of appendixes elaborate on various details and calculated related to the main text.

II. SINGLE TIME-SCALE APPROXIMATION

At conventional pure critical points, scaling emerges directly from the existence of a RG fixed point. Moreover, in an epsilon expansion, the *scaling functions* are obtained to leading order by a simple matching procedure (REF). The *static* equilibrium FRG for the random elastic problem is, aside from the complication of a functional fixed point, very similar to such an ordinary RG calculation. The important distinction is the nonanalyticity of the $T=0$ fixed point function $\Delta^*(u)$ which at finite temperature results in a narrow boundary layer for small $u \lesssim \tilde{T}_l \epsilon$, whose width continuously decreases under the FRG as the running effective temperature $\tilde{T}_l$ (see below) flows to zero. The corresponding growth of the mean-squared curvature of the effective potential felt by the manifold is a hint of unconventional behavior not present in ordinary critical theories.

The influence of this divergence is very dramatic in the dynamics. In this section, we attempt to naively extend the conventional RG approach to calculating response functions to the random manifold problem. This conventional approach implicitly assumes the existence of a single time scale (at a given wave vector), as we shall soon see. This assumption leads to somewhat unsatisfactory results for the response function, forcing us to reconsider the distribution of time-scales in Sec. III. Although we will ultimately conclude that the single time-scale calculation is fundamentally incorrect, it is useful to review the methodology of this approach.

We begin by reviewing the basics of the FRG. We employ the Martin-Siggia-Rose (MSR) formalism [34], in order to use field-theoretic techniques. The MSR approach is based on the generating functional $Z[h,\hat{h}]$ for the disorder-averaged correlation and response functions

$$Z[h,\hat{h}] = \int DuD\hat{u}e^{-S[u,\hat{u}]+\int_{rt}\hat{h}_{rt}u_{rt}+h_{rt}i\hat{u}_{rt}}, \tag{10}$$

where the dynamics in Eq. (1) is encoded in the action $S[u,\hat{u}]=S_0[u,\hat{u}]+S_{\text{int}}[u,\hat{u}]$, with

$$S_0[u,\hat{u}] = \int_{rt} i\hat{u}_{rt}(\overline{\eta}\partial_t - \nabla^2)u_{rt} - \overline{\eta}T\int_{rt}(i\hat{u}_{rt})(i\hat{u}_{rt}), \tag{11}$$

$$S_{\text{int}}[u,\hat{u}] = -\frac{1}{2}\int_{rtt'}(i\hat{u}_{rt})(i\hat{u}_{rt'})\Delta(u_{rt}-u_{rt'}), \tag{12}$$

where $\hat{h}$, h are source fields, and we have put a overbar on the friction coefficient $\eta\to\overline{\eta}$ to indicate the mean, i.e., that it is for now a constant uniform in space. As justified below we have set $c=1$. We use the Ito convention to regularize equal-time response functions, i.e., $R_q(t,t)=0$ and $R_q(t^+,t)=1/\overline{\eta}$. The disorder averaged response and correlations are given by

$$R_q(t,t') = \overline{\frac{\delta\langle u_q(t)\rangle}{\delta h_q(t')}} = \langle u_q(t)i\hat{u}_{-q}(t')\rangle_S, \tag{13}$$

$$C_q(t,t') = \overline{\langle u_q(t)u_{-q}(t')\rangle} = \langle u_q(t)u_{-q}(t')\rangle_S. \tag{14}$$

The FRG in its Wilsonian formulation begins by introducing an ultraviolet (short distance) cutoff Λ on the spatial Fourier wave vectors. In the FRG, this cutoff is progressively reduced to $\Lambda_l=\Lambda e^{-l}$ $(0<l<\infty)$. At each stage of the RG, the spatial Fourier components of u, $\hat{u}$ are divided into "slow" and "fast" modes, with momenta in the range $0<k<\Lambda_l e^{-dl}$ and $\Lambda_l e^{-dl}<k<\Lambda_l$, respectively. The fast modes are then integrated out, working perturbatively in S_{int} to one loop order, and l is increased by dl. This leads, in the limit of infinitesimal rescaling $dl\to 0$, to a smooth renormalization of the effective action for the remaining slow modes, and hence to continuous FRG equations for the running disorder correlator $\Delta_l(u)$.

Naive power counting (see, e.g., Refs. [16,17]) indicates that all terms beyond those in Eqs. (11), (12) are irrelevant, so we for the moment neglect their generation under the FRG. The flow of the random pinning correlator $\Delta_l(u)$ has been derived many times previously [18–20,22]. It is better expressed in terms of the dimensionless rescaled pinning force correlator $\tilde{\Delta}_l(u)$ defined such that

$$\Delta_l(u) = \frac{\Lambda^\epsilon}{A_d}e^{-\epsilon l}e^{2\zeta l}\tilde{\Delta}_l(ue^{-\zeta l}), \tag{15}$$

with $A_d=S_d/(2\pi)^d=1/[2^{d-1}\pi^{d/2}\Gamma(d/2)]$, and reads

$$\partial_l\tilde{\Delta}(u) = (\epsilon-2\zeta)\tilde{\Delta}(u) + \zeta u\tilde{\Delta}'(u) + \tilde{T}_l\tilde{\Delta}''(u) \tag{16}$$

$$+ \tilde{\Delta}''(u)[\tilde{\Delta}(0)-\tilde{\Delta}(u)] - \tilde{\Delta}'(u)^2. \tag{17}$$

Here the fluctuation dissipation theorem ensures that the temperature $T_l=T$ is uncorrected but the effective dimensionless temperature $\tilde{T}_l=A_dT\Lambda^{d-2}e^{-\theta l}$ itself flows to zero, controlled by the energy fluctuation exponent $\theta=d-2+2\zeta$, the temperature being formally irrelevant. Here and in the following we do not not make any spatial or temporal rescalings of coordinates or momenta.

Study of the one loop FRG equation shows that, with the proper value for the roughness exponent $\zeta\sim O(\epsilon)$ depending on the universality class (RB, RF, or RP), the dimensionless disorder correlator converges *nonuniformly* to a nonanalytic "fixed-point" function $\Delta^*(u)$ formally of order $\sim O(\epsilon)$ as $l\to\infty$, whose functional form is not important for this discussion (see, however, Sec. IV for much more details). The nonuniformity of this convergence is due to a boundary-layer centered on $u=0$, whose width decreases continuously with scale [18,22]. In particular one can show that, to this order [18,22]

$$\lim_{l\to+\infty}\tilde{T}_l\tilde{\Delta}_l''(0) \to -\chi^2. \tag{18}$$

Thus asymptotically the curvature of the correlator diverges with the scale [here $\chi=|\Delta'^*(0^+)|$] is a constant depending of the universality class, e.g., for periodic systems $\chi=\tilde{\chi}\epsilon$, $\epsilon=4-d$.

To one loop, all that remains is the renormalization of the mean friction coefficient, since the elastic modulus is fixed by Galilean invariance [17,18,22] and the temperature by the fluctuation-dissipation-theorem (FDT). This was determined in Refs. [18–20,22]:

$$\partial_l\overline{\eta} = \Gamma_l\overline{\eta}, \tag{19}$$

where $\Gamma_l=-\tilde{\Delta}_l''(0)\sim_{l\to+\infty}\chi^2/\tilde{T}_l$ thus grows with the scale as

$$\Gamma_l \sim \tilde{\beta}e^{\theta l}, \tag{20}$$

$$\tilde{\beta} = T^*/T, \tag{21}$$

where $\tilde{\beta}$ is the reduced bare inverse temperature and $T^*=\chi^2\Lambda^{2-d}/A_d$ a nonuniversal temperature scale. Equation (19) implies *activated scaling* [18,22] since the friction coefficient, which plays the role of a time scale $\tau=\eta/\Lambda^2$, grows exponentially with the length e^l:

$$\overline{\eta_l}=\overline{\eta_0}\exp\left[\frac{\tilde{\beta}}{\theta}(e^{\theta l}-1)\right]. \tag{22}$$

[in $d=4$ one has $\Gamma_l=\tilde{\beta}e^{2l}/l^2$ and $\tilde{\beta}=\tilde{\chi}^2\Lambda^{-2}/(TA_4)$].

Activated dynamics leads to ambiguities in a single time-scale approach, as can be seen from a simple matching argument. We consider for simplicity the equilibrium dynamics, in which the response and correlation functions are time-translationally invariant (TTI). It is then convenient to work in terms of both frequency ω and wave vector. The usual RG considerations lead one naively to expect that the response function obeys the relation

$$R_k(\omega)=e^{2l}R_{ke^l}(\omega\tau_l), \tag{23}$$

where $\tau_l=e^{2l}\overline{\eta_l}/\overline{\eta_0}$. We now obtain two *inequivalent* scaling forms by matching. In particular, if we choose $ke^l=1$ (we set $\Lambda=1$ for now), we find

$$R_k^{(1)}(\omega)=\frac{1}{k^2}\mathcal{R}^{(1)}(\omega\tau_k), \tag{24}$$

with

$$\tau_k=\frac{1}{k^2}e^{(\tilde{\beta}/\theta)(k^{-\theta}-1)}. \tag{25}$$

If, however, we choose $\omega\tau_l=1$, we find asymptotically

$$R_k^{(2)}=\left[\frac{1}{\tilde{\beta}}\ln\left(\frac{1}{\omega}\right)\right]^{2/\theta}\mathcal{R}^{(2)}\left(\frac{1}{\tilde{\beta}}k^{\theta}|\ln\omega|\right). \tag{26}$$

Note that these two forms *cannot* be related by redefining the two scaling functions $\mathcal{R}^{(1/2)}$, as can be seen from the fact that $\ln(\omega\tau_k)\sim\ln\omega+(\chi/\theta)k^{-\theta}=(\chi/\theta+k^{\theta}|\ln\omega|)/k^{\theta}$, which is *not* a function of $k^{\theta}|\ln\omega|$ only.

The inequivalence of Eqs. (24), (26) appears to point to the existence of two scaling regimes, which we will call the X and Y scaling limits. The first is formally defined by defining the scaling variable $Y=\omega\tau_k$. With Y fixed and ω, $k\to 0$, one obtains the scaling regime in Eq. (24). The second scaling regime obtains with $X=k^{\theta}|\ln\omega|$ fixed and ω, $k\to 0$. To reconcile the two regimes, note that $\ln Y\sim(\chi/\theta+X)/k^{\theta}$, so that for fixed X, as ω, $k\to 0$ (the X scaling limit) $Y\to\infty$. The second scaling regime [Eq. (26)] thus appears to occur at the "boundary" $(Y=\infty)$ of the first.

We have indeed verified this behavior, and obtained the analytical scaling forms similar to those in Refs. [24,26] using FRG techniques *under the assumption of a single characteristic time scale parametrized by* $\overline{\eta}$. These calculations are performed in Appendix A for equilibrium and in Appendix B for the more general nonequilibrium situation. Interestingly, the general equations for the response function in this approach share some similarities to those arising in the (infinitely connected and/or large N) mean field limit of a number of model glasses. As discussed in Appendices A and B their solutions exhibit several time regimes with various aging scaling forms and also show differences compared to the mean field.

Many features of these results, however, point to problems with the single time scale assumption, as also discussed in Appendix A. The real-time response function is found to be an *increasing* function of wave vector at fixed time in the logarithmic (X) scaling regime. This somewhat unexpected (and possibly unphysical) behavior is apparently a very general consequence of the mere *existence* of two distinct scaling limits, and hence is inevitable given the single time scale approach. More significantly, the appearance of a sharply defined τ_k in the *mean* response function (in the Y regime) is difficult to understand on physical grounds. Even in (random) models involving only a small number of degrees of freedom, while a given sample may be characterized by a longest relaxation time, the sample to sample variations of this would generally lead, as espoused in the Introduction, to the disappearance of such a time in the mean response. In the collective elastic model considered here, interactions between the enormous number of modes with differing wave vector (and hence differing relaxation rates) would only worsen the situation.

III. BROAD DISTRIBUTIONS OF TIME SCALES: SIMPLIFIED APPROACH

A. Distribution of relaxation times and the *f* term

Up to this point, we have assumed that the dynamics at each scale can be described by a single friction coefficient $\tau_l=\overline{\eta_l}$, which corresponds to a sharply defined time scale for relaxation. On general grounds, however, we should expect extremely broad distributions of relaxation times. This follows simply from the Arrhenius law

$$\tau_k=\tau_{l=\ln(k/\Lambda)}\sim\tau_0\exp(U_k/T), \tag{27}$$

which estimates the time required to overcome an energy barrier of height U_k at scale k. At low temperature, even a modestly wide distribution of U_k gives rise to extremely broadly distributed τ_k. If this distribution is sufficiently broad, it is no longer adequately characterized by its average, and indeed many physical quantities may depend upon the precise form of the distribution.

We now investigate how this distribution can be incorporated into the FRG treatment within the MSR formalism. We will consider a spatially varying friction coefficient $\eta(r)$, with the equation of motion (1) modified to

$$\eta(r)\partial_t u_{rt}=c\nabla_r^2u_{rt}+f(u_{rt},r)+\zeta(r,t), \tag{28}$$

where, in order to maintain the stationary equilibrium Boltzmann probability distribution function, the noise correlations are modified to

$$\langle\zeta(r,t)\zeta(r',t')\rangle=2\eta(r)T\delta^d(r-r')\delta(t-t'). \tag{29}$$

For simplicity, we will initially take $\eta(r)$ to be identically and independently distributed at each r, according to the distribution $P(\eta)$. The distribution naturally enters the MSR theory via its characteristic function, which we parametrize by $F(z)$:

$$\int_0^{+\infty} d\eta P(\eta) e^{-z\eta} = e^{-F[z]}. \tag{30}$$

The Taylor series expansion of $F(z)$ thereby gives the connected cumulants of η,

$$F[z] = \sum_{m=1}^{\infty} \eta^{(m)} \frac{(-1)^{m+1}}{m!} z^m, \tag{31}$$

where $\eta^{(m)} = [\overline{\eta^m}]_C$ is the mth cumulant (connected) moment of η. For the continuum field $\eta(r)$, the analogous expression is

$$\overline{\exp\left(-\int_r \eta(r) z(r)\right)} = \exp\left(-\int_r F[z(r)]\right). \tag{32}$$

We initially assume no cross correlations between $\eta(r)$ and $f(u,r)$, though these can to some extent be generated in perturbation theory. The single time scale model studied in the previous section (and Appendices A and B) with $\eta(r)=\overline{\eta}$ corresponds to $F(z)=\overline{\eta}z$. Although this is not essential, we shall assume here an initially narrow (but not δ function) distribution $P(\eta)$. As shown in Appendix C, even if initially $F=\overline{\eta}z$, a higher-order analysis shows that a nontrivial distribution is generated under coarse graining.

In the MSR formalism, the modified equation of motion (28), (29) is described by the action

$$S_0[u,\hat{u}] = \int_{rt} \overline{\eta}\, i\hat{u}_{rt}\partial_t u_{rt} - i\hat{u}_{rt} c \nabla_r^2 u_{rt} - \overline{\eta} T i\hat{u}_{rt} i\hat{u}_{rt}, \tag{33}$$

$$S_{\rm int}[u,\hat{u}] = \int_r \tilde{F}\left[\int_t (i\hat{u}_{rt}\partial_t u_{rt} - T i\hat{u}_{rt} i\hat{u}_{rt})\right] - \frac{1}{2}\int_{r,t,t'} (i\hat{u}_{rt})(i\hat{u}_{rt'})\Delta(u_{rt}-u_{rt'}). \tag{34}$$

We have defined $F[z]=\overline{\eta}z+\tilde{F}[z]$ where $\tilde{F}[z]$ starts with higher powers of z, to do perturbation theory using the average friction coefficient. One immediate remark is that the statistical tilt/translational symmetry (STS) holds [27], thus c will not be corrected and so we set it to $c=1$. Note that the $\overline{\eta}$ kinetic term could equally well be considered as an interaction term, in the spirit of a "perturbation theory in $i\omega$" with bare propagator simply $R_{q,\omega}=1/q^2$ [in real time $R_q(t,t')=1/q^2\delta(t^- - t')$]. Note also that for each realization, the instantaneous response function satisfies

$$R(\mathbf{r},\mathbf{r}',t-t'=0+) = \frac{1}{\eta(r)}\delta(\mathbf{r}-\mathbf{r}'). \tag{35}$$

Averaging over disorder gives $R_k(t-t'=0^+)=\overline{1/\eta}$.

Although it may seem obvious, it is important to stress at this stage that the renormalized relaxation time moments are measurable quantities. One can define the renormalized moments in the usual way from the effective action Γ. Taking the same form as Eq. (34), one has

$$\Gamma_{\rm int} = -\frac{\eta^{(2)}}{2}\int_{rtt'} \hat{u}_{rt}\dot{u}_{rt}\hat{u}_{rt'}\dot{u}_{rt} + \cdots, \tag{36}$$

with of course many higher order terms describing the higher friction coefficient moments, momentum dependence of vertices, etc. As usual in field theory, correlation functions are exactly evaluated at tree level using this effective action—thus the $\eta^{(2)}$ term here has the meaning of a fully renormalized second moment on the scale of the system size (or infrared momentum cutoff). One may then consider the physically defined relaxation time from Eq. (8) and construct its second moment

$$\overline{\langle t\rangle_L^2} = \frac{1}{L^{2(d+2)}}\int dt dt' t t' \int_{r_1 r_1' r_2' r_2} \overline{\langle u_{r_1 t}\hat{u}_{r_1' 0}\rangle\langle u_{r_2 t'}\hat{u}_{r_2' 0}\rangle}. \tag{37}$$

On physical grounds, in equilibrium, we expect that the latter product of response functions in two "replicas" is the same as considering the product of two subsequent responses in a single replica, provided the two response measurements are taken far apart:

$$\overline{\langle u_{r_1 t}\hat{u}_{r_1' 0}\rangle\langle u_{r_2 t'}\hat{u}_{r_2' 0}\rangle} = \lim_{\tau\to\infty}\overline{\langle u_{r_1 t+\tau}\hat{u}_{r_1'\tau}u_{r_2 t'}\hat{u}_{r_2' 0}\rangle}. \tag{38}$$

This latter four-point function can be calculated using the effective action above. One finds

$$\lim_{\tau\to\infty}\int_{r_1 r_1' r_2' r_2}\overline{\langle u_{r_1 t+\tau}\hat{u}_{r_1'\tau}u_{r_2 t'}\hat{u}_{r_2' 0}\rangle} = L^{2d} R_{q_0 t} R_{q_0 t'} + \eta^{(2)} L^d (R_{q_0} * \dot{R}_{q_0})_t (R_{q_0} * \dot{R}_{q_0})_{t'}, \tag{39}$$

where $q_0 \sim 1/L$ is the infrared momentum cutoff and the asterisk denotes the convolution in the time domain. Integrating over the time coordinates and using the result of statistical translational symmetry $R(q_0,0)=1/q_0^2$, one then obtains

$$\overline{\langle t\rangle_L^2} \sim (\overline{\eta}L^2)^2 + \eta^{(2)} L^{4-d}. \tag{40}$$

For a not too broad distribution of friction coefficients with scale independent $\eta^{(2)}$, the correction due to $\eta^{(2)}$ is vanishing for $d>4$, and small compared to the first "disconnected" term (scaling as $\overline{\langle t\rangle_L}^2$) for any d. However, for the glassy dynamics studied here, we will find $\eta^{(2)}$ is *exponentially* larger than $\overline{\eta}^2$ as a function of L, so that in fact the second term is dominant. Thus the second moment of the physical relaxation time $\overline{\langle t\rangle_L^2}$ indeed measures the coupling constant $\eta^{(2)}$ as promised.

B. No pinning disorder: the random friction model

We now turn to the FRG analysis of the modified action in Eqs. (33), (34). We first consider *only* the effects of randomness in η, *neglecting* the pinning disorder Δ. This defines a *random friction* model described by the MSR action with $\Delta=0$. Remarkably, the random friction model represents an infinite manifold of fixed points parametrized by $F[z]$. Indeed, a diagrammatic treatment explicitly shows the absence of renormalization of $F[z]$ order by order in $\tilde{F}$. Despite this

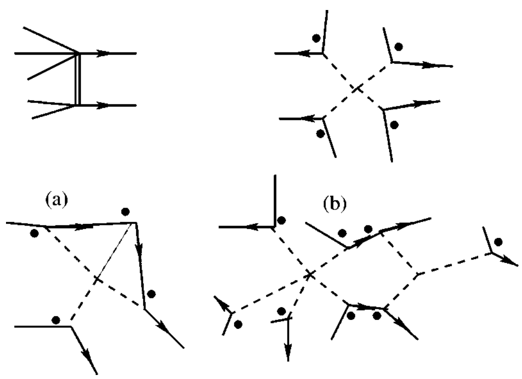

FIG. 1. Top: graphical representation of the disorder vertex (double lines) and of the F term vertex (the dots represent the time derivatives) where the arrows represent a $\hat{u}$ response field and solid line a u field (arrows are along inreasing time). Bottom: (a) corrections of the F term to itself at $T=0$ which vanish (b) corrections to order F^2 which also vanish (as all orders do—see text).

absence of renormalization, the random friction model represents a nontrivial interacting field theory.

For simplicity, we sketch this here for $T=0$. In this limit, the vertex is

$$\int_r \tilde{F}\left[\int_t i\hat{u}_{rt}\partial_t u_{rt}\right]. \tag{41}$$

Diagrams occuring in the expansion of F are indicated in Figs. 1(a) and 1(b). The fields connected by dotted lines occur at the same spatial point, solid lines with and without arrows indicate u and $\hat{u}$ fields, respectively. Considering a product of the form

$$\int_{r_1}\left(\int_{t_1} i\hat{u}_{r_1t_1}\partial_{t_1}u_{r_1t_1}\right)^{n_1}\int_{r_2}\left(\int_{t_2} i\hat{u}_{r_2t_2}\partial_{t_2}u_{r_2t_2}\right)^{n_2}, \tag{42}$$

the only possible contractions contain products of the type $\langle\partial_{t_j}u_{r_jt_j}i\hat{u}_{r_kt_k}\rangle$ with no time loop allowed. Thus all relative time integrals factor and one is left with products of integrals of the type $\int_t\partial_t R(r_j-r_k,t)$ which vanish since the response function vanishes for $t<0$ and $t\to+\infty$ [35]. Thus F does not correct F. However, F itself produces new terms such as

$$\int_{r,t} i\hat{u}_{rt}\partial_t^2 u_{rt}, \tag{43}$$

$$\int_{r,t_1,t_2} i\hat{u}_{rt_1}\partial_{t_1}^2 u_{rt_1} i\hat{u}_{rt_2}\partial_{t_2}^2 u_{rt_2}, \tag{44}$$

obtained by time gradient expansions, with nonvanishing coefficients [of the form $\sim\int_t t\partial_t R(r_j-\tau_k,t)$], as well as similar terms with higher order time derivatives (note that similar terms containing also higher order spatial gradients are also generated, but we will not consider them as important here [36]). One can embed these new terms into a new function

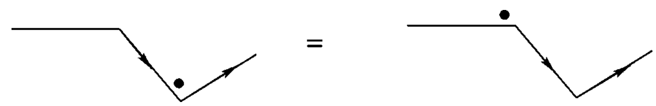

FIG. 2. Shifting of a time derivative from an internal line (in the middle) to an external one (right) along a line of response functions at $T=0$ (works for disorder as well as F vertices).

$$\int_r F_2\left[\int_t i\hat{u}_{rt}\partial_t^2 u_{rt}\right], \tag{45}$$

and so on—the full systematics of these new terms will be examined later in Appendix E and Sec. IV. It is important to note for consistency that there is also no feedback from higher-derivative terms such as F_2 back into F. Graphically, as in Figs. 1(a) and 1(b), one can perform time integration by parts along each line joining several vertices which leads to terms $\int_t\partial_t^p\hat{u}_t\partial_t^q u_t$ with $p+q>1$. Indeed a very useful rule is represented graphically on Fig. 2. One can simply shift the time derivative along any internal response line to the external one (at $T=0$ any diagram is a tree of such lines) since, schematically,

$$u_t\langle i\hat{u}_t\partial_{t_1}u_{t_1}\rangle i\hat{u}_{t_1} = u_t\partial_{t_1}R_{t_1,t}i\hat{u}_{t_1} \tag{46}$$

$$=-u_t\partial_t R_{t_1,t}i\hat{u}_{t_1}\to\partial_t u_t R_{t_1,t}i\hat{u}_{t_1}, \tag{47}$$

after integration by parts. In the Fourier domain, this rule is just conservation of frequency along all solid lines, since the interactions are all fully nonlocal in time and therefore do not carry frequency.

To conclude, the apparent nonrenormalization of $F[z]$ makes it tempting to define *a manifold of fixed point theories* indexed by $F[z]$. These fixed points are quite interesting and nontrivial. For instance, the computation of the averaged response function at $T=0$ can be mapped exactly onto the problem of calculating the partition function for a self-avoiding walk. This is developed further in Appendix D.

C. Pinning disorder: distribution of barriers

We now consider the combined effects of the pinning disorder and distribution of time scales. Because the pinning disorder can be defined in a purely static theory (using the equilibrium Bolzmann partition function), its renormalization is unaffected by the F term. However, the converse is not correct. Due to the nonrenormalization of F in the random friction model, we must consider only terms of order $F^p\Delta^q$, with $q\geq 1$. The leading nonvanishing terms correcting F at $O(\Delta)$ are linear in F, and are indicated diagrammatically in Figs. 3, 4. They are computed in detail in Appendix C but one easily sees the structure of the result, thanks to the property of shifting internal time derivatives (dots in the figures) to the external ones, e.g., that the three graphs in Fig. 3 have identical values.

There is a subtle distinction between the contributions in Fig. 3 and those in Fig. 4. In particular, in the diagram of Fig. 4, the pinning vertex suffers contractions between both of its independent time variables (i.e., graphically both ends of the

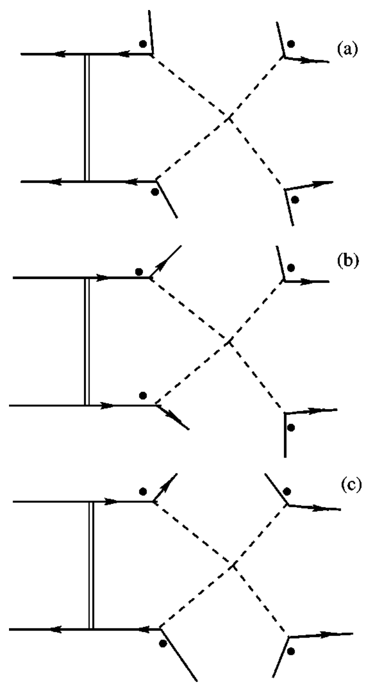

FIG. 3. Graphs involving pinning disorder which correct the F term proportionally to F''.

double line) and *the same* time variable (dotted leg) of the f-term vertex. The locality of the response function therefore implies that the two internal times of the pinning force correlator are constrained to be nearby, justifying a temporal gradient expansion and hence giving a leading contribution proportional to $\Delta''(0)$. In the diagrams of Fig. 3, by contrast, the two times of the pinning vertex are contracted with *different* legs of the F term. These diagrams therefore generate in fact more general terms involving $\Delta''(u_t - u_{t'})$ with free integration over t and t'. If $|u_t - u_{t'}|$ is not extremely small (within the boundary layer), this is a small correction lacking the singular temperature dependence. It is thus not clear at this stage whether or not the graphs in Fig. 3 should in fact be interpreted as renormalizations of the F term.

In fact, the true situation is more delicate, and will be returned to in Sec. IV. For the moment, however, we will shut our eyes to this complication, and gain some physical insight by taking into account both sets of diagrams as renormalizations of the f term. Their sum, integrated in the momentum shell, gives the following correction to F:

$$\delta F[z] = -\Delta''(0) S_d \Lambda_l^{d-4} dl (zF'[z] + 2z^2 F''[z]). \quad (48)$$

Here the F' and F'' terms comes from the diagrams in Figs. 4, 3, respectively. In agreement with Eq. (19) it can be rewritten as

$$\partial_l F_l[z] = (\partial_l \ln \overline{\eta_l})(zF_l'[z] + 2z^2 F_l''[z]). \quad (49)$$

Note that the disputed diagrams in Fig. 3 do not contribute to the mean relaxation time due to the second derivative of z, so that $\overline{\eta}$ is unambiguous. The RG equations for the connected moments $\eta^{(n)}$ of η are thereby obtained using Eq. (31) as

$$\eta_l^{(n)} \sim \eta_0^{(n)} \left(\frac{\overline{\eta_l}}{\overline{\eta_0}} \right)^{2n^2 - n}. \quad (50)$$

It is more convenient and physical to introduce the random barrier $U = \ln \eta$, and the barrier corresponding to the average relaxation time $U_l = \ln \overline{\eta_l}$. Changing to the energy variable $v = \ln z$, and letting $G_l(v) = F_l(e^v)$ gives

$$\partial_{U_l} G_l[v] = 2G_l''[v] - G_l'[v], \quad (51)$$

i.e., a diffusion with drift equation. Some physical understanding of $G(v)$ can be obtained from the two extreme limits

$$G(v) \sim \begin{cases} \overline{\eta} e^v, & v \to -\infty, \\ v - \ln P(0), & v \to \infty, \end{cases} \quad (52)$$

as can easily be found from Eq. (30), assuming a constant probability density for small barriers $0 < P(0) < \infty$. More generally, using Eq. (30) the diffusing and drifting "density" G_l is related to the barrier probability distribution via

$$G_l[v] = -\ln \int dU P_l(U) e^{-e^{U+v}}. \quad (53)$$

Formally, the solution of Eq. (51) is given by

$$G_l(v) = \int dw \frac{1}{\sqrt{8\pi U_l}} \exp\left[-\frac{(v - U_l - w)^2}{8U_l} \right] G_0(w), \quad (54)$$

and inverting the results via Eq. (53) to obtain $P_l(U)$. A simple approximation may be applied in the regime of large $U \gg U_l$ and $U_l \gg 1$, in which $G_l(v) \ll 1$. In this case, it is valid (and justified *a posteriori* since the distribution of barriers become broad) to replace $e^{-e^{U+v}}$ by $\theta(U < -v)$ and thus one gets that $P_l(U) \approx G_l'(-U)$. This yields [via Eq. (54) or directly differentiating Eq. (51)]

$$P_l(U) \approx \frac{1}{\sqrt{8\pi U_l}} \exp\left(-\frac{(U + U_l)^2}{8U_l} \right), \quad U \gtrsim U_l. \quad (55)$$

Note that this asymptotic form reproduces all cumulants $\eta_l^{(n)} = \overline{[\exp(nU)]}_C \sim \exp[(2n^2 - n)U_l]$ as expected.

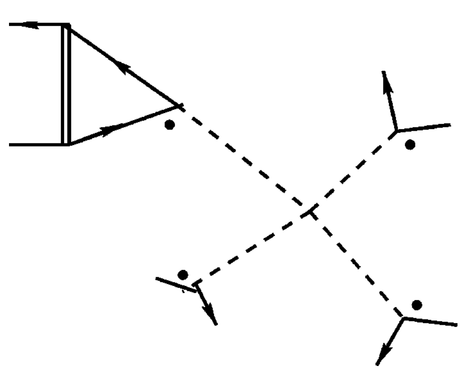

FIG. 4. Graphs involving pinning disorder which correct the F term proportionally to F'.

Equation (55) is clearly not exact. Indeed, a breakdown of Eq. (55) is inevitable on physical grounds, since the mean/typical barrier cannot be negative. It suggests a distribution of barriers with a width proportional to $\sqrt{U_l}$, and hence a peaked distribution (since the mean barrier $\propto U_l$). Nevertheless, it does represent a very broad (in fact log-normal) renormalized distribution of characteristic times η. While there is no reason to believe that such a log-normal tail is exact, the true distribution of relaxation times will certainly be very broad, with significant consequences for the average response functions.

D. Breakdown of $\omega\tau_k$ scaling

The first consequences of this broad distribution occur in the variance $\eta^{(2)}$ of the relaxation time, and hence at $O(\omega^2)$ in the response function. We therefore examine more carefully the $O(\omega^2)$ terms in the dynamical action, but for the moment still neglecting the full functional dependence of these terms (i.e., on $u_t - u_{t'}$). In the kinetic part of the action (representing relaxation times and their fluctuations), we include the following terms:

$$S_{\text{kin}} = \int_{rt} [\overline{\eta} i\hat{u}_{rt}\partial_t u_{rt} + D i\hat{u}_{rt}\partial_t^2 u_{rt}] - \frac{\eta^{(2)}}{2}\int_{rt_1t_2} i\hat{u}_{rt_1}\partial_{t_1}u_{rt_1} i\hat{u}_{rt_2}\partial_{t_2}u_{rt_2}. \tag{56}$$

One has in general, defining $\langle t^n\rangle_R = \int_t t^n R_k(t)/\int_t R_k(t)$:

$$\langle t\rangle_R = \overline{\eta}, \tag{57}$$

$$\langle t^2\rangle_R - \langle t\rangle_R^2 = \overline{\eta^2} - 2D. \tag{58}$$

There is no generic constraint on the sign of D. If the inverse response function contained only the two above terms ($\overline{\eta}$ and D), then causality requires D to be positive (similar to an inertial term) [37]. Since in general these are only trucation of an infinite series of terms in power of $i\omega$, the only constraint is causality, i.e., that all poles in ω lie on the same side of the real axis. These three couplings satisfy the following closed RG flow equations to first order in Δ:

$$\partial_l D_l = \Gamma_l D_l - \Gamma_l \Lambda_l^{-2}\overline{\eta_l^2} - A_d \Lambda_l^{d-2}\eta_l^{(2)}, \tag{59}$$

$$\partial_l \overline{\eta_l} = \Gamma_l \overline{\eta_l}, \tag{60}$$

$$\partial_l \eta_l^{(2)} = 6\Gamma_l \eta_l^{(2)}, \tag{61}$$

where $\Gamma_l = -\tilde{\Delta}_l''(0) \sim \tilde{\beta} e^{\theta l}$ and the correction to D from $\eta^{(2)}$ is the graph represented in Fig. 5.

For $\eta^{(2)} = 0$ the equations for D_l and η_l, which can be obtained, e.g., from Eqs. (A2), (A6) by expansion to second order in $i\omega$, are consistently solved with $D_l \sim \Lambda_l^{-2}\overline{\eta_l^2}$, in the limit of large l, where $\overline{\eta_l} = \overline{\eta_0}\exp[\tilde{\beta}(e^{\theta l}-1)/\theta]$, consistent with the Taylor expansion of the putative scaling function $g(y = i\omega\tau_k)$ given in (A9) based on the single time scale analysis (A9).

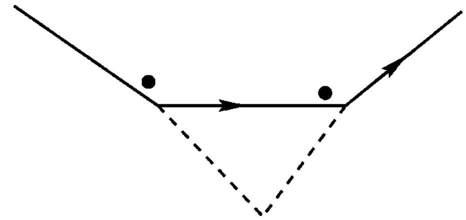

FIG. 5. Correction to the $(i\omega)^2$ term in the response function coming from the second moment $\eta^{(2)}$ of the relaxation time distribution.

For $\eta^{(2)} > 0$, however, the broad distribution of relaxation times completely alters the situation. From the above equations $\eta_l^{(2)} \sim \eta_0^{(2)}(\overline{\eta_l}/\overline{\eta_0})^6$ and, hence, $\eta_l^{(2)} \gg \overline{\eta_l^2}$ (the mere exponential prefactors are negligible) at large l. Thus the feedback of $\eta^{(2)}$ in D dominates the renormalization of D, and at large l one finds

$$D_l \sim \frac{A_d \Lambda_l^{d-2}}{6\Gamma_l}\eta_0^{(2)}\left(\frac{\overline{\eta_l}}{\overline{\eta_0}}\right)^6. \tag{62}$$

Thus, allowing for fluctuations in relaxation times invalidates the $\omega\tau_k$ scaling form already at order ω^2.

It is still possible, within the approximation scheme of the present Section, to obtain an equation for the disorder averaged response function. This is explored further in Appendix E.

IV. DISTRIBUTION OF TIME SCALES: FULL STRUCTURE OF THE DYNAMICAL FIELD THEORY

We have established the mechanism for breakdown of the unphysical $\omega\tau_k$ scaling regime and described the indications of a broad distribution of timescales within the FRG. However, to properly determine this distribution and its consequences, e.g., on the mean response function, requires a much more complete analysis. While we have unfortunately so far been unable to carry this program to completion, in this section we will detail the formal structure within which this analysis must be carried out. In particular, we shall see that the distribution of relaxation times and its consequences is encoded within the boundary layer (BL) regime already present in the statics. An understanding of the equilibrium dynamics is therefore contingent first upon an understanding of the static BL, and we first describe the rather complex structure therein. Following this discussion, we show how the BL regime recurs in dynamical theory, and show how it can be formulated to describe broad distributions and nontrivial scaling of the moments of the relaxation time.

A. Statics thermal boundary layer

In the appropriate limit the dynamical theory should reproduce the results for the corresponding statics quantity. We can therefore benefit from the knowledge of the thermal boundary layer in the statics. To do so let us review how the static disorder correlations are encoded in the dynamical formalism.

At equilibrium, the part of the dynamical action containing the static disorder correlations comprises those terms with no explicit time derivatives, and reads

$$S_{\rm int} = -\frac{1}{2}\int_{rt_1t_2} i\hat{u}_{rt_1} i\hat{u}_{rt_2}\Delta(u_{rt_1}-u_{rt_2}) - \frac{1}{6}\int_{rt_1t_2t_3} i\hat{u}_{rt_1} i\hat{u}_{rt_2} i\hat{u}_{rt_3} S_d^{(3)}(u_{rt_1},u_{rt_2},u_{rt_3}) - \cdots . \tag{63}$$

This form is easily understood as arising from the cumulants of the pinning force. The relation was given for the second cumulant in Eq. (3) and for higher ones it reads

$$\overline{f(u_1,r_1)\cdots f(u_k,r_k)}^{\,c} = (-)^k S_d^{(k)}(u_1,\ldots,u_k)\,\delta^d(r_1,\ldots,r_k), \tag{64}$$

with $S_d^{(2)}(u,u')\equiv\Delta(u-u')$ as in Eq. (3). Due to statistical translational invariance $S^{(k)}(u_1,\ldots,u_k)=S^{(k)}(u_1+v,\ldots,u_k+v)$ and satisfy reflection symmetry $S^{(k)}(-u_1,\ldots,-u_k)=(-)^k S^{(k)}(u_1,\ldots,u_k)$. The cumulants higher than second are generated by coarse graining, and are thus included here from the start.

The static problem being defined from the equilibrium Boltzmann measure [see Eq. (6)], deals not with the distribution of the random force but with that of the random potential

$$\overline{V(u_1,r_1)\cdots V(u_k,r_k)}^{\,c} = (-)^k S^{(k)}(u_1,\ldots,u_k)\,\delta^d(r_1,\ldots,r_k). \tag{65}$$

Since $f(u,r)=-\partial_u V(u,r)$ one has

$$\Delta(u) = -R''(u), \tag{66}$$

$$S_d^{(3)}(u_1,u_2,u_3) = \partial_1\partial_2\partial_3 S^{(3)}(u_1,u_2,u_3), \tag{67}$$

and so on.

It is straighforward to derive the one loop FRG equation in the Wilson scheme for these cumulants using the dynamical formulation. They are conveniently expressed using rescaled cumulants

$$S_d^{(k)}[u_{a_1},\ldots,u_{a_k}] = A_d^{1-k}\Lambda_l^{d+k(\zeta-\theta)}\tilde{S}_d^{(k)}[u_{a_1}\Lambda_l^{\zeta},\ldots,u_{a_k}\Lambda_l^{\zeta}],$$

and read for the second and third cumulant

$$\partial_l\tilde{\Delta}(u) = (\epsilon-2\zeta+\zeta u\partial_u)\tilde{\Delta}(u) + \tilde{T}_l\tilde{\Delta}''(u) + 2\tilde{S}_{100}(0,u,0) - \tilde{\Delta}'(u)^2 - \tilde{\Delta}''(u)[\tilde{\Delta}(u)-\tilde{\Delta}(0)], \tag{68}$$

$$\begin{aligned}\partial_l\tilde{S}(u_1,u_2,u_3) &= (-2+2\epsilon-3\zeta+\zeta u_i\partial_{u_i})\tilde{S}(u_1,u_2,u_3) + \tfrac{1}{2}\tilde{T}_l[\tilde{S}_{200}(u_1,u_2,u_3)+\tilde{S}_{020}(u_1,u_2,u_3)+\tilde{S}_{002}(u_1,u_2,u_3)] - \tfrac{1}{24}\tilde{\Delta}(0)\\ &\times\Big[\tilde{S}_{200}(u_1,u_2,u_3)+\tilde{S}_{020}(u_1,u_2,u_3)+\tilde{S}_{002}(u_1,u_2,u_3)\Big] - \tfrac{1}{4}\tilde{\Delta}(u_1-u_2)\tilde{S}_{110}(u_1,u_2,u_3) - \tfrac{1}{4}\tilde{\Delta}''(u_1-u_2)\Big[\tilde{S}(u_1,u_1,u_3)\\ &-\tilde{S}(u_1,u_2,u_3)\Big] - \tfrac{1}{4}\tilde{\Delta}'(u_1-u_2)\Big[\tilde{S}_{010}(u_1,u_2,u_3)-\tilde{S}_{100}(u_1,u_2,u_3)+\tilde{S}_{010}(u_1,u_1,u_3)+\tilde{S}_{100}(u_1,u_1,u_3)\Big],\end{aligned} \tag{69}$$

where we have denoted $S_d^{(3)}=S$ and we have suppressed explicitly the feedback of the fourth cumulant $S_d^{(4)}$ into the third one. One can check that this gives exactly the derivatives (67) of the one loop FRG equations for the static correlators R and $S^{(3)}$ displayed in Eqs. (6), (7) in Ref. [24]. These relations (67) should indeed be preserved by RG at equilibrium.

As discussed in Ref. [23] when all arguments of these functions are distinct and order one conventional scaling holds. That is, at large scales for which $T_l\to 0$, the functions $\tilde{S}^{(k)}$ approach well-defined nonanalytic fixed point forms $\tilde{S}^{(k)*}$. Moreover, these can be naively organized in an $\epsilon=4-d$ expansion in which $\tilde{S}^{(k)*}\sim\epsilon^k$, $k\geqslant 3$. Naively this would allow the truncation of the hierarchy of FRG equations for the $\tilde{S}^{(k)}$, neglecting feedback of the $k>p$ cumulants with an accuracy of $O(\epsilon^p)$. However, the convergence to these values is highly nonuniform as mentioned in Sec. II since at nonzero temperature these functions remain analytic at $u=0$. A detailed analysis of the static hierarchy of FRG equations relating these cumulants revealed the existence of a thermal boundary layer (TBL) of the form

$$\tilde{\Delta}(u) = \tilde{\Delta}*(0) - \tilde{T}_l f(\tilde{u}), \tag{70}$$

$$\tilde{u} = \epsilon\tilde{\chi}u/\tilde{T}_l, \tag{71}$$

for $\tilde{u}=O(1)$, $\tilde{T}_l\ll\epsilon^2$ and f an analytic function with $f(x)\sim|x|$ at large x to match the cusp of the zero temperature solution. For higher cumulants the very unconventional TBL scaling implies that it is no longer legitimate to neglect the feedback of higher cumulants (the nth cumulants gets a feedback from the n and $n+1$ ones). Therefore, we are unable to truncate and solve the hierarchy of FRG equations. Instead, in Ref. [23] we argued for the consistency of a thermal boundary layer ansatz (TBLA), which for the force cumulants reads

$$\tilde{S}_d^{(k)}(u_1,\ldots u_k) = \begin{cases} f_k + (\tilde{\chi}\epsilon)^{k-2}T_l s_d^{(k)}(\tilde{u}_1,\ldots,\tilde{u}_k), & k \text{ even},\\ (\tilde{\chi}\epsilon)^{k-2}T_l s_d^{(k)}(\tilde{u}_1,\ldots,\tilde{u}_k), & k \text{ odd},\end{cases} \tag{72}$$

where s_d are well defined functions of order one in the TBL $\tilde{u}\sim 1$. The set of (l-dependent) constants

$$f_{2p} = \tilde{S}_d^{(2p)}(0, \ldots, 0)/(\tilde{\chi}\epsilon)^{2p}, \tag{73}$$

with $f_2 = \tilde{\Delta}(0)/(\tilde{\chi}\epsilon)^2$, have the meaning of the linearized random force cumulants within the zero temperature Larkin description. As discussed in Ref. [23] the crucial difference with the naive dimensional reduction result, where the f_{2p} are unrenormalized, is that they get feedback from the TBL functions and acquire nontrivial asymptotic values.

The TBLA encodes a huge amount of physics—in particular, all the distributions of minima degeneracy responsible for large thermal fluctuations in the droplet picture, as detailed in Ref. [23]. For instance, averages such as Eq. (7) can be estimated using the TBLA, the coefficients c_n being in principle determined by the functions $s^{(k)}$. This already nontrivial structure must now be generalized to intrinsically dynamical quantities.

B. Dynamical hierarchy of kinetic coefficients

In a conventional dynamical renormalization group in the MSR formalism a succession of individual terms are added to the action corresponding to increasingly high frequency kinetic coefficients, e.g., for a particle the Stokes drag, inertial mass, … . For the disordered elastic manifold, however, we recognize that these kinetic coefficients have a broad distribution characterized by an infinite set of cumulants and cross correlations, which moreover can be nontrivial functions of displacement field differences. The latter dependence was neglected in the approximate treatment of Sec. III. The need for treating it was already indicated in the ambiguities in the diagrams of Fig. 3. Each of these cumulants and cross correlations appears as a distinct interaction *function* in the MSR action.

By symmetry (time translation and STS, statistical reflection, causality) alone, the set of all such interactions contributing to the effective action at zero temperature can be written as

$$S = \sum_{n=1}^{\infty} \sum_{P=\{p_i^k\}} \int_{rt_1\cdots t_n} i\hat{u}_{rt_1} \cdots i\hat{u}_{rt_n},$$

$$S_P^{(n)}(u_{rt_1}, \cdots, u_{rt_n}) \prod_{k=1}^{+\infty} \prod_{i=1}^{n} (\partial_{t_i}^k u_{rt_i})^{p_i^k}, \tag{74}$$

where $p_i^k \geq 0$ and from STS symmetry, $S(u_1+u, \ldots, u_n+u) = S(u_1, \ldots, u_n)$ are translationnally invariant, and statistical reflection implies the full action is also invariant under $(\hat{u}, u) \to (-\hat{u}, -u)$. The random force correlators correspond to

$$S_{P=0}^{(n)}(u_1, \ldots, u_n) = S_d^{(n)}(u_1, \ldots, u_n), \tag{75}$$

where $P=0$ above indicates the function with $p_i^k=0$ for all i, k. Other terms correspond to intrinsically dynamical cumulants.

It is instructive to begin the characterization of such terms at $T=0$ by listing all possible forms in order of increasing number m of time derivatives and order of cumulant (i.e., the number of independent times which equals the number of $\hat{u}$ fields at $T=0$). Each term in Eq. (74) can be assigned $m = \sum_{i=1}^n \sum_{k=1}^{+\infty} k p_i^k$. For organisational purposes it is convenient to rewrite the action of Eq. (74) in a schematic (but transparent) notation, first expanding in number of cumulants:

$$S = i\hat{u}_1[k^2 + \Sigma(\partial_1)]u_1 + S_{\text{int}}, \tag{76}$$

$$\Sigma(s) = \overline{\eta} s + D s^2 + \cdots, \tag{77}$$

$$S_{\text{int}} = -\tfrac{1}{2} i\hat{u}_1 i\hat{u}_2 \mathcal{S}_{12} - \tfrac{1}{6} i\hat{u}_1 i\hat{u}_2 i\hat{u}_3 \mathcal{S}_{123} - \cdots, \tag{78}$$

with $s = i\omega$. Here the subscripts 1, 2, …, refer to different times being independently integrated over in the action at same space point r (further integrated on). The $\mathcal{S}_{1,2,\ldots}$ are then functions of the $u_1, u_2, \ldots$, and their time derivatives. We then expand each of these in increasing number m of time derivatives

$$\mathcal{S}_{12} = \Delta(u_{12}) + (\dot{u}_1 - \dot{u}_2)G(u_{12}) + \dot{u}_1\dot{u}_2 A(u_{12}) + (\dot{u}_1^2 + \dot{u}_2^2)B(u_{12}) + (\ddot{u}_1 - \ddot{u}_2)C(u_{12}) + \cdots, \tag{79}$$

$$\mathcal{S}_{123} = S(u_1,u_2,u_3) + \tfrac{1}{3}\big[\dot{u}_1 H(u_1;u_2,u_3) + \dot{u}_2 H(u_2;u_3,u_1) + \dot{u}_3 H(u_3;u_1,u_2)\big] + \dot{u}_1\dot{u}_2 W(u_1,u_2;u_3) + \cdots \tag{80}$$

$$\cdots . \tag{81}$$

As discussed above, each new term in Eqs. (79), (81) corresponds to statistical properties of the random kinetic coefficients and forces in a renormalized equation of motion, in particular,

$$\cdots + D(u,r)\ddot{u} + \eta(u,r)\dot{u} = \nabla^2 u + f(u,r) + g(u,r)\dot{u}^2 + \cdots + \zeta(r,t), \tag{82}$$

with

$$\overline{D(u,r)} = D, \quad \overline{\eta(u,r) f(u',r')}^{\,c} = -G(u-u')\delta(r-r'), \quad \overline{\eta(u,r)\eta(u',r')}^{\,c} = A(u-u')\delta(r-r'), \tag{83}$$

$$\overline{g(u,r) f(u',r')}^{\,c} = B(u-u')\delta(r-r'), \quad \overline{f(u,r) D(u',r')}^{\,c} = C(u-u')\delta(r-r'), \tag{84}$$

$$\overline{\eta(u_1,r_1) f(u_2,r_2) f(u_3,r_3)}^{\,c} = \tfrac{1}{3} H(u_1;u_2,u_3)\delta(r_1-r_2)\delta(r_2-r_3), \tag{85}$$

$$\overline{\eta(u_1,r_1)\eta(u_2,r_2) f(u_3,r_3)}^{\,c} = \tfrac{1}{3} W(u_1,u_2;u_3). \tag{86}$$

In the approximate treatment of Sec. III, $\eta^{(2)}$ hence corresponds to $A(u)$ approximated as $A(0)$. Note that it is the small argument behavior of $A(u)$ (and its higher cumulant analogs) that is related to the physically interesting second (higher) relaxation time moment $\eta^{(2)} \sim \overline{\langle t \rangle^{2c}}$ $(\eta^{(n)} \sim \overline{\langle t \rangle^{nc}})$. Hence these relaxation times are encoded within the BL regime of these functions. This was also apparent from the naïve renormalization of η by $\Delta''(0)$, also a BL quantity. We will return to the problem of the dynamic BLs in $G(u), A(u), \ldots$, momentarily.

Although it is convenient as above for the purpose of *enumerating* terms in the dynamical action to first separate by cumulant index n and then by number of time derivatives m, conceptually we analyze them in the opposite scheme, i.e., collecting all terms of a given m, and organizing these afterward in order of n. This scheme is clearly convenient insofar as the first term $(m=0)$ of each of the $\mathcal{S}_{12\cdots n}$ corresponds to the nth term of the static cumulant hierarchy, so that the set of terms with $m=0$ satisfies a closed hierarchy of FRG equations independent of those with $m>0$. We now demonstrate diagrammatically that, at zero temperature, a similar property holds for $m>0$. In particular, all terms of any given m will satisfy a closed containing only terms with $m' \leqslant m$. Thus, one may imagine (dream of?) solving the FRG equations up to level m, then using this solution to complete a closed set of FRG equations for level $m+1$, and iteratively solving for higher and higher m.

This closure relies on the rule of conservation of powers of frequency, established in Sec. III. Recall that this occurs because at $T=0$, the correlation function vanishes, and all contractions take the form of *causal* response functions. Thus no closed time loops can appear in any diagram. This implies that internal time derivatives which appear in any diagram appear as factors of frequency of some external leg to which they are connected. In any case, this rule implies that, since all terms in the action have $m \geqslant 0$, terms with $m' > m$ can never reduce their number of time derivatives by contraction with another vertex at $T=0$, and hence cannot renormalize m vertices. This is true to all orders for diagrams with any number of loops. The frequency conservation rule implies in fact a more detailed result. If the quadratic terms in $\Sigma(s)$ $(\eta, D, \ldots)$ are regarded themselves as coupling constants [38] (with $m=1,2,\ldots$), then each term in the FRG equation for any quantity at level m is a product of factors for which the total frequency level (i.e., Σm_i for all terms i in the product) is exactly m. Thus a static quantity (e.g., Δ) can renormalize a dynamic one (e.g., G, η) only in combination (i.e., multiplied by) another dynamic quantity, and so on.

One can also establish a set of rules to understand how cumulants with different n are connected in the FRG equations. At $T=0$, this process is highly constrained, since each contraction involves one response function, which removes one $\hat{u}$, it is straightforward to count the possible connections. We will restrict our attention to one loop diagrams, anticipating future nonperturbative exploration using the exact RG [23,39], in which only these appear (and in any case only these are consistently treated in the Wilsonian scheme of this paper). The counting is illustrated for such one loop diagrams in Figs. 6, 7. One readily sees that when N vertices are combined in this manner, the resulting vertex which is renormalized in the effective action contains a total number of independent times (or $\hat{u}$ factors) $n=\Sigma_{i=1}^{N} n_i - N$, due to the N response functions appearing in the loop.

With these rules in mind, we can describe the structure of the FRG hierarchy as far as the feeding of terms of a given m, n into other m', n'. We note symbolically by S_m^n the terms

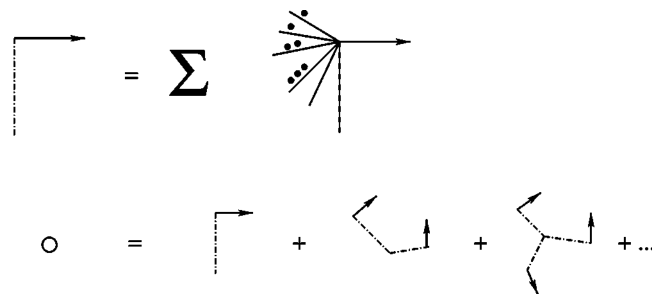

FIG. 6. Compact notation for a generic vertex at $T=0$.

with n response fields and m time derivatives. The term $\overline{\eta}$ is S_1^1, the response function is the quadratic part of S_m^1 [we note $R^{-1}=\text{quad}(S_m^1)$] and the cumulants $\eta^{(n)}$ are included in S_n^n. From the above discussion, neglecting rescaling terms, the structure of the FRG equations reads

$$\delta S_m^{(n)} = S_m^{(n+1)} + \sum_{m'=0}^{m} \sum_{n'=1}^{n+1} S_{m'}^{(n')} S_{m-m'}^{(n+2-n')} + \sum_{m'+m''\leqslant m} \sum_{n'+n''\leqslant n+2} S_{m'}^{(n')} S_{m''}^{(n'')} S_{m-m'-m''}^{(n+3-n'-n'')} + \cdots . \tag{87}$$

It is straightforward to see that this series contains a finite number of terms for any given m, n. Let us suppose that an N-loop term exists, such that each vertex making it up has n_i time integrations. Suppose of these N vertices, $n_i>1$ for N' of them and $n_i=1$ for the remaining $N-N'$. Then $n=\Sigma_{i=1}^{N} n_i - N = N - N' + \Sigma_{i=1}^{N'} n_i - N = \Sigma_{i=1}^{N'}(n_i - 1)$. Hence at most $N' \leqslant n$. Now the remaining $N-N' \geqslant N-n$ vertices have only one time integration. Since there are no allowed local terms with-

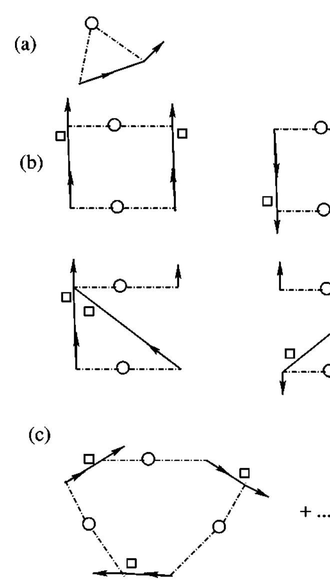

FIG. 7. One loop diagrams which correct the effective action at $T=0$: the internal lines contain the full response function and the graphs are 1P irreducible. Graph (a) is a "tadpole." Graphs (b) and (c) (and higher orders) correct terms with $n \geqslant 1,2,3$, respectively.

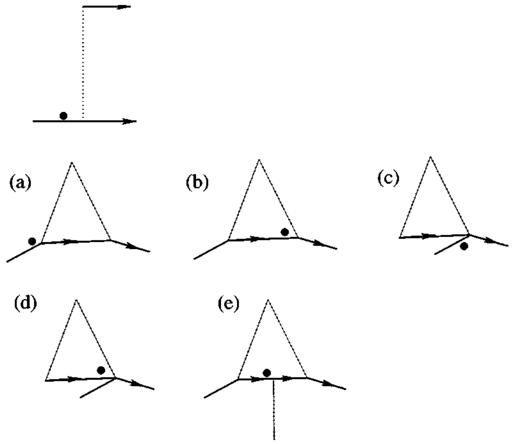

FIG. 8. Shown is the G vertex (top image with no alphabetic label) and diagrammatic corrections to $\overline{\eta}$. Graphs (a)–(d) are contributions from tadpoles of the G vertex [note that (a) and (c) cancel by the same mechanism as dimensional reduction, and that (d) vanishes upon integration by parts on internal line]. Graph (e) is the contribution from $\overline{\eta}\Delta$.

out time derivatives, these must each have $m_i \geqslant 1$, i.e., $m \geqslant N-n$. Turning this around, $N \leqslant m+n$, so that the series of one loop diagrams terminates at at most $(m+n)$th order. Clearly from Eq. (87), each order contains a finite number of terms, so that the one loop FRG equations are finite.

1. Terms proportional to frequency m=1

We will now examine level $m=1$ and $m=2$ of the hierarchy. For $m=1$ we will restrict to study the FRG equation for terms with $n \leqslant 2$ for which we need terms up to $n=3$:

$$S_1^{(1-3)} = \int_{rt} \overline{\eta} i\hat{u}_{rt}\dot{u}_{rt} - \frac{1}{2}\int_{rt_1t_2} i\hat{u}_{rt_1} i\hat{u}_{rt_2}(\dot{u}_{rt_1} - \dot{u}_{rt_2})G(u_{rt_1} - u_{rt_2}) - \frac{1}{6}\int_{rt_1t_2t_3} i\hat{u}_{rt_1} i\hat{u}_{rt_2} i\hat{u}_{rt_3}\dot{u}_{rt_1} H(u_{rt_1}, u_{rt_2}, u_{rt_3}), \tag{88}$$

where $G(-u) = -G(u)$.

The renormalization of $\overline{\eta}$ and G is determined by a standard if cumbersome one loop calculation performed in the Appendix F. The corresponding graphs are represented in Figs. 8, 9. From dimensional analysis and the structural form of the FRG equation (87) we see that and G and H as single frequency $m=1$ terms will be fed by $O(\overline{\eta}\Delta^2)$ and $O(\overline{\eta}\Delta^3)$, respectively. Hence, given the rapid growth of $\overline{\eta}$ with scale, we expect these functions to be at least growing as $\overline{\eta}$ with scale. We thus defined rescaled functions

$$G_l(u) = \overline{\eta}_l \frac{\Lambda_l^{2-d} e^{\zeta l}}{A_d} \tilde{G}(u e^{-\zeta l}), \tag{89}$$

$$H_l(u_1, u_2, u_3) = \overline{\eta}_l \frac{\Lambda_l^{4-2d} e^{2\zeta l}}{A_d^2} \tilde{H}(u_1 e^{-\zeta l}, u_2 e^{-\zeta l}, u_3 e^{-\zeta l}),$$

in terms of which one finds the flow equation (40) for $\overline{\eta}$ and $\tilde{G}(u)$:

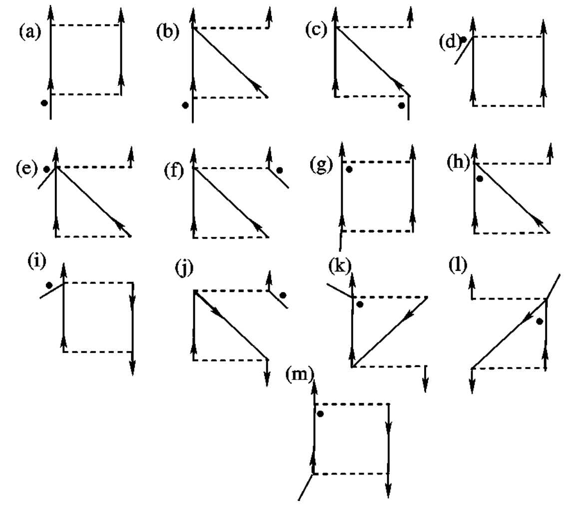

FIG. 9. Corrections to G.

$$\partial_l \overline{\eta} = [\tilde{G}'(0) - \tilde{\Delta}''(0)]\overline{\eta}, \tag{90}$$

$$\partial_l \tilde{G} = (-2 + \epsilon - \zeta)\tilde{G} + \zeta u \partial_u \tilde{G} - 2\tilde{\Delta}''\tilde{G} + [\tilde{\Delta}(0) - \tilde{\Delta}]\tilde{G}'' - 3\tilde{\Delta}'\tilde{G}' - \tilde{G}'(0)\tilde{G} - \tilde{G}'(0)\tilde{\Delta}' + \tilde{\Delta}'[2\tilde{\Delta}''(0) + 2\tilde{\Delta}''] + \tilde{S}_{110}(0,0,u) + \tfrac{1}{3}[\tilde{H}_{010}(u,0,0) - 2\tilde{H}_{001}(0,u,0) - \tilde{H}_{100}(0,u,0)]. \tag{91}$$

Because of the above rescaling (89) no explicit $\overline{\eta}$ appear in Eq. (91).

Since $\tilde{G}'(0)$ appears on the same footing as $\tilde{\Delta}''(0) \sim 1/\tilde{T}_l$ in Eq. (90) it is natural to expect it to grow unboundedly with scale in the same fashion. Indeed inspection of Eq. (91) reveals that $\tilde{G}$ is fed by a term explicitly proportional to $\tilde{\Delta}''(0)$ which can consistently be balanced by the $\tilde{G}'(0)$ term appearing in the first line of the same equation. Therefore we are led to expect that $\tilde{G}$ itself, similar to $\tilde{\Delta}$, should exhibit a thermal boundary layer and the effect of temperature will be essential in understanding the structure properly. We will come back after taking a brief look at the $m=2$ terms.

2. Terms proportional to square frequency m=2

The $m=2$ terms, restricting to $(n \leqslant 2)$-th order cumulant read

$$S_2^{(1-2)} = \int_{rt} i\hat{u}_{rt} D\ddot{u}_{rt} - \frac{1}{2}\int_{rt_1t_2} i\hat{u}_{rt_1} i\hat{u}_{rt_2}[\dot{u}_{rt_1}\dot{u}_{rt_2} A(u_{rt_1} - u_{rt_2}) + (\dot{u}_{rt_1}^2 + \dot{u}_{rt_2}^2)B(u_{rt_1} - u_{rt_2}) + (\ddot{u}_{rt_1} - \ddot{u}_{rt_2})C(u_{rt_1} - u_{rt_2})]. \tag{92}$$

The renormalization of D and of the functions A, B, and C via a one loop calculation is performed in the appendixes. It turns out to be convenient to define

$$B_1(u) = B(u) - \tfrac{1}{2}C'(u), \tag{93}$$

which simplifies the equations, the physics being explained below. We define rescaled quantities as follows:

$$D = \Lambda_l^{-2}\overline{D}, \tag{94}$$

$$A(u) = \frac{\Lambda_l^{-d}}{A_d}\tilde{A}(ue^{-\zeta l}), \quad B_1(u) = \frac{\Lambda_l^{-d}}{A_d}\tilde{B}_1(ue^{-\zeta l}),$$

$$C(u) = \frac{\Lambda_l^{-d}e^{\zeta l}}{A_d}\tilde{C}(ue^{-\zeta l}), \tag{95}$$

$$W_l(u_1,u_2,u_3) = \overline{\eta}_l^2\frac{\Lambda_l^{2-2d}e^{\zeta l}}{A_d^2}\tilde{W}(u_1e^{-\zeta l},u_2e^{-\zeta l},u_3e^{-\zeta l}),$$

and one finds

$$\partial_l\tilde{D} = [-2 - \tilde{\Delta}''(0)]\tilde{D} - \tilde{A}(0) + \tilde{C}'(0) - \overline{\eta}^2[2\overline{G'}(0) - \tilde{\Delta}''(0)], \tag{96}$$

together with the FRG coupled flow equations for $\tilde{A}(u)$, $\tilde{B}(u)$, and $\tilde{C}(u)$ which read

$$\partial_l\tilde{A} = [-d + \zeta u\partial_u - 2\tilde{\Delta}''(0) - 4\tilde{\Delta}'']\tilde{A} - 4\tilde{\Delta}'\tilde{A}' + [\tilde{\Delta}(0) - \tilde{\Delta}]\tilde{A}'' + \overline{\eta}^2[2\tilde{G}'(0)\tilde{G}' + 5\tilde{G}'^2 - 4\tilde{G}'\tilde{\Delta}''(0) - 8\tilde{G}'\tilde{\Delta}'' + 2\tilde{\Delta}''^2 + 4\tilde{G}\tilde{G}'' - 4\tilde{\Delta}'\tilde{G}''], \tag{97}$$

$$\partial_l\tilde{B}_1 = \{-d + \zeta u\partial_u - 3[\tilde{\Delta}'' + \tilde{\Delta}''(0)]\}\tilde{B}_1 + 3\tilde{\Delta}''\tilde{B}_1(0) - 4\tilde{\Delta}'\tilde{B}_1' + [\tilde{\Delta}(0) - \tilde{\Delta}]\tilde{B}_1' + \tilde{A}(0)\tilde{\Delta}'' + \overline{\eta}^2[-\tilde{G}'(0)\tilde{G}' - \tilde{G}'^2 - \tilde{G}\tilde{G}'' + 2\tilde{G}'\tilde{\Delta}''(0) + \tilde{G}'(0)\tilde{\Delta}'' + \tilde{G}'\tilde{\Delta}'' + \tilde{\Delta}'\tilde{G}'' - \tilde{\Delta}''(0)\tilde{\Delta}''], \tag{98}$$

$$\partial_l\tilde{C} = [-\zeta - d + \zeta u\partial_u - \tilde{\Delta}''(0) - 2\tilde{\Delta}'']\tilde{C} + \tilde{\Delta}'\tilde{A}(0) - \tilde{\Delta}'\tilde{C}'(0) - 3\tilde{\Delta}'\tilde{C}' + \left[\tilde{\Delta}(0) - \tilde{\Delta}\right]\tilde{C}'' + 2\tilde{D}\tilde{\Delta}'[\tilde{\Delta}''(0) + \tilde{\Delta}''] + \overline{\eta}^2\Big[-2\tilde{G}\tilde{G}'(0) + 4\tilde{\Delta}'\tilde{G}'(0) - 2\tilde{G}\tilde{G}' + 2\tilde{\Delta}'\tilde{G}' + 2\tilde{G}\tilde{\Delta}''(0) - 3\tilde{\Delta}'\tilde{\Delta}''(0) + 2\tilde{G}\tilde{\Delta}'' - 2\tilde{\Delta}'\tilde{\Delta}''\Big]. \tag{99}$$

From these equations we expect exponential growth of $\tilde{D}$, $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ at least as fast as $\overline{\eta}^2$ due to the feeding terms. We expect from the considerations of Sec. III that the growth is actually faster. Indeed considering the A flow equation at the origin gives $\partial_l\tilde{A}(0) = -6\tilde{\Delta}''(0)\tilde{A}(0)$ keeping the largest terms of order $1/\tilde{T}_l$ and neglecting feeding terms. Note the similarity to the result of Sec. III. While we expect this qualitative behavior, the precise nature of the growth is more subtle due to the fact that at nonzero temperature $A(0)$ no longer satisfies a closed equation. We will discuss this in more details below.

Let us close this section by noticing that all nontrivial terms in the right-hand side of the above set of FRG equations (β functions) for G, A, B, and C cancel when one chooses

$$\tilde{G}(u) = \tilde{\Delta}'(u), \tag{100}$$

$$\tilde{A}(u) = -\overline{\eta}^2\tilde{\Delta}''(u), \tag{101}$$

$$\tilde{C} = \tilde{D}\tilde{\Delta}'(u), \tag{102}$$

$$\tilde{B}_1 = \tilde{D}\tfrac{1}{2}\tilde{\Delta}''(u), \tag{103}$$

$$\tilde{H}(u_1,u_2,u_3) = 3\tilde{S}_{100}(u_1,u_2,u_3), \tag{104}$$

$$\tilde{W}(u_1,u_2,u_3) = -3\tilde{S}_{110}(u_1,u_2,u_3), \tag{105}$$

and furthermore the flow of the kinetic coefficients simplify to

$$\partial_l\overline{\eta} = 0, \tag{106}$$

$$\partial_l D = 0. \tag{107}$$

This remarkable property, which serves as a useful check on the RG equations, can be understood in terms of a simple integrable model which is studied in the Appendix G (which has very different physics from the one studied here).

C. Dynamical thermal boundary layer

1. Dynamical action at non-zero temperature and FDT

At $T > 0$ two new effects must be taken into account not present at $T=0$. First, in addition to the kinetic coefficients studied above one must take into account a variety of thermal noise terms. In the dynamical action this corresponds to terms with two or more $\hat{u}$ field having the same time argument. Second, new thermal contractions are possible using the nonzero correlation function $\langle uu\rangle$ of the Gaussian theory. We first focus on the former, and discuss the additional contractions at the end of this subsubsection. The action takes the general form

$$S = \sum_{n=1}^{\infty}\sum_{P=\{p_i^k\},R=\{r_i^k\}}\int_{rt_1\cdots t_n} i\hat{u}_{rt_1}\cdots i\hat{u}_{rt_n}S_{P,R}^{(n)} \times(u_{rt_1},\ldots,u_{rt_n})\prod_{k=0}^{+\infty}\prod_{i=1}^{n}(\partial_{t_i}^k i\hat{u}_{rt_i})^{r_i^k}(\partial_{t_i}^k u_{rt_i})^{p_i^k}, \tag{108}$$

with $p_i^0=0$ and the $r_i^k \geq 0$ are integers. The $S_{P,R}^{(n)}$ are translationnally invariant functions. Compared to the $T=0$ action it has additional powers of $\hat{u}_i$ and possibly their time derivatives (such vertices are shown in Fig. 10). There is a temperature homogeneity degree $s=\sum_k\sum_i r_i^k$ such that the term is $\sim T^s$. The standard thermal noise corresponds to $S_{P=0,R}^{(1)} = -\eta T$, with $r_i^k = 2\delta_{k0}\delta_{i1}$.

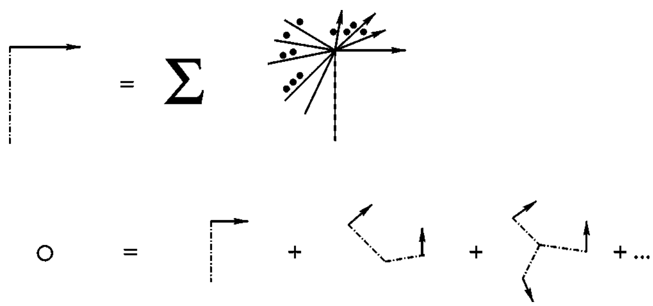

FIG. 10. Compact notation for a generic vertex at $T>0$.

In order to maintain the fluctuation dissipation theorem (FDT) there are relations between these vertices. A useful symmetry which constrains the allowed form of these terms is

$$i\hat{u}_{rt} \to i\hat{u}_{r,-t} - \lambda_r \dot{u}_{r,-t}, \tag{109}$$

$$u_{rt} \to u_{r,-t}, \tag{110}$$

[we mean $\dot{u}_{-t}=u'(-t)$]. For actions with no explicit time dependence, such as considered here, one can then later make a change of variables $t\to -t$ in integrals over times. We apply this to the bare action (11), (12). Consider first the infinitesimal variation of the interaction term

$$\delta S_{\rm int} = -\int_r \lambda_r \int_{t_1,t_2} i\hat{u}_{rt_1}\dot{u}_{rt_2}\Delta(u_{rt_1}-u_{rt_2}) + O(\lambda^2)$$
$$= -\int_r \lambda_r \int_{t_1,t_2} i\hat{u}_{rt_1}\partial_{t_2}R'(u_{rt_1}-u_{rt_2}) + O(\lambda^2), \tag{111}$$

using $\Delta(u)=-R''(u)$, a consequence of potentiality. This integrates to a boundary term which is a function only of the coordinates u and corresponds to the energy difference between the initial and final times (see below). Hence the interaction term is invariant for an arbitrary (r-dependent) λ_r. Unfortunately, this large symmetry is quadratically broken by Eq. (11). First, the variation of the elastic term vanishes (up to boundary terms) only for spatially constant $\lambda_r=\lambda$. Thus the full action for $\eta=0$ has a continuous global λ symmetry (109). This can be used, e.g., to put constraints on the terms appearing in the FRG equation order by order in η [41]. More importantly, the remaining terms in the action are only invariant under a discrete transformation, specifically Eq. (109), with

$$\lambda = \frac{1}{T}. \tag{112}$$

Note that they are, however, *exactly* invariant (no boundary term).

Having established the invariance of the bare model under the symmetry (112) we know that it should be preserved under renormalization. We must thus understand the consequences of this symmetry for a more general effective action. The relation to FDT is apparent since, performing the transformation in the path integral defining the response function one finds

$$R_{t_2-t_1} = \langle i\hat{u}_{t_1}u_{t_2}\rangle, \tag{113}$$

$$= \langle (i\hat{u}_{-t_1} - \dot{u}_{-t_1})\dot{u}_{-t_2}\rangle = R_{t_1-t_2} + \frac{\dot{C}_{t_1-t_2}}{T}, \tag{114}$$

i.e., the FDT relation for two point functions. The same is obtained from considering the action of the symmetry (112) on a general form (i.e., nonlocal in time) for the quadratic part of the effective action functional [42,50].

We now discuss more precisely the conditions on the boundary terms which relate this symmetry to the FDT. This is simplest to see first in the context of the theory before averaging over disorder. Let us define the Ito path integral

$$Z(u_f,t_f;u_i,t_i) = \int_{u(t_i)=u_i}^{u(t_f)=u_f} D\hat{u}Du e^{-S_v}, \tag{115}$$

$$\int du_f Z(u_f,t_f;u_i,t_i) = 1, \tag{116}$$

the (normalized) conditional probability to find the system in state u_f at time t_f given that it is in state u_i at time t_i. Here S_V is the MSR dynamical action in a given disorder realization. By construction the Boltzmann measure is the stationary distribution for this $Z(u_f,t_f;u_i,t_i)$ regarded as an evolution operator

$$\int du_f\, Z(u_f,t_f;u_i,t_i)e^{-[H(u_i)-H(u_f)]/T} = 1. \tag{117}$$

Note that if under the transformation above $S_V\to S_V+\delta S_V$ where $\delta S_V=\delta S_V[u_i,u_f]$ is a function only of u_i and u_f (boundary term) one has

$$Z(u_f,t_f;u_i,t_i) = Z(u_i,t_f;u_f,t_i)e^{-\delta S_V}, \tag{118}$$

since t is changed to $-t$ and thus boundary times t_i and t_f must be interchanged. Interchanging u_i and u_f in Eq. (117) and using Eq. (118) the normalization condition (116) is found to hold only if

$$\delta S_V[u_i,u_f] = \frac{1}{T}[H(u_{t_f}) - H(u_{t_i})]. \tag{119}$$

If, on the other hand, δS_V depends also on the time derivatives at the boundary, then the FDT may or may not be satisfied depending on the precise nature of the boundary terms [41].

Upon averaging over disorder $\overline{e^{S_V+\delta S_V}}^V = e^{S+\delta S}$, the shift δS obtained after transformation on the disorder averaged MSR action. It is readily seen that for the bare action, δS is a sum of a single time integral cross correlation boundary term (and one response field) and a term with no time integral representing the second cumulant of the random portential $V(u)$. More generally, if one performs the transformation (109) on the coarse grained model, one must obtain a δS which is a sum of boundary terms, each containing less response fields than time integrals. The nth cumulant of the renormalized static disorder can then be retrieved from the corresponding boundary term with no time integral and $1/T^n$ factor.

We are now prepared to discuss one how can construct the effective action at finite temperature taking into account the constraints from the FDT. It is useful to note that from the fundamental fields $i\hat{u}$ and u two linear combinations transform simply under the symmetry (112)

$$\dot{u} \to -\dot{u}, \tag{120}$$

$$Y = 2Ti\hat{u} - \dot{u} \to 2Ti\hat{u} - \dot{u}. \tag{121}$$

Terms in the effective action which are exactly invariant (i.e., whose variation do not produce any boundary terms) must involve $\hat{u}$ only in the combination $2Ti\hat{u}-\dot{u}$. Examples will be constructed below.

It is clear from these considerations that nonzero T terms can have time derivatives replaced by $T\hat{u}$. Therefore it is natural to group terms which formerly (at $T=0$) were organized by the frequency power m by the more general index

$$M = N_{\hat{u}} - n + m, \tag{122}$$

where $N_{\hat{u}}$ is the number of $\hat{u}$ fields appearing in the term and n the number of independent times, as before. Terms with a given M, n can mix under the FDT transformation (112).

Let us start with $M=1$ and $n=2$. The only possible combination of the above invariants is, using symbolic notations as above:

$$S_{T,1}^{(2)} = \frac{(2Ti\hat{u}_1 - \dot{u}_1)^2 - \dot{u}_1^2}{8T} i\hat{u}_2 G(u_1 - u_2) + (1 \leftrightarrow 2)$$
$$= -\frac{1}{2} i\hat{u}_1 i\hat{u}_2 [(\dot{u}_1 - \dot{u}_2) - T(i\hat{u}_1 - i\hat{u}_2)] G(u_1 - u_2), \tag{123}$$

recovering the zero temperature G term together with a nonzero T partner term. Note that at time t_1 we have used the invariant combination Y^2, taking care to remove the piece proportional to $\dot{u}_1^2$ since there must be at least one response field at every time. This is not possible whenever there is an odd total number of fields $\dot{u}$ and $\hat{u}$ at any particular time. Thus the $i\hat{u}_2$ above cannot be embedded in a term exactly invariant. Hence $G(u)$ must be a total derivative and the above term gives a nonvanishing boundary variation under the transformation (112). This term can also be understood before disorder averaging. It corresponds to replacing the naive zero T dynamical term $\eta(u,r) i\hat{u}\dot{u}$ corresponding to the damping in Eq. (82) by the invariant combination

$$\int_{rt} \eta(u_{rt}, r) \frac{[\dot{u}_{rt}^2 - (2Ti\hat{u}_{rt} - \dot{u}_{rt})^2]}{4T}. \tag{124}$$

Note that expanding out the factor in this term demonstrates that including u dependence in the damping coefficient has given rise to a white thermal noise which [for $D(u,r)=g(u,r)=0$] has u-dependent variance

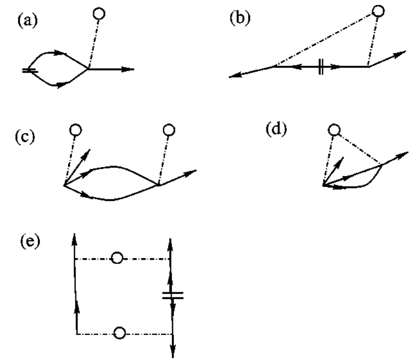

FIG. 11. One loop diagrams which correct the effective action at $T>0$ (in addition to the one for $T=0$): the internal lines contain the full response function and the graphs are 1P irreducible. (a), (b) The tadpoles using the full correlation function [the only possible ones as (d) should not be counted as it is a two loop diagram]. (c) The generic new one loop diagram at $T>0$ with two vertices. (e) One example of expanding the full correlation.

$$\langle \zeta(r,t)\zeta(r',t')\rangle = 2\eta(u,r)T\delta(t-t')\delta(r-r'). \tag{125}$$

The fact that $G(u)$ is a total derivative then follows simply from its interpretation as a cross cumulant of $\eta(u,r)$ and the conservative random pinning force $f(u,r)$. Based on this reasoning it is clear that the function $B(u)$ and $C(u)$ must also be total derivatives.

Note that the finite T partner of the G term is generated in parallel to it from graphs of the form (e) in Fig. 11. One easily checks that it corrects the temperature term by $\delta\eta T\hat{u}\hat{u}$ where $\delta\eta$ corresponds to the correction (90) from $G'(0)$ so as to maintain FDT relation.

Let us now examine the terms $M=2$ and $n=2$. One can write the possible terms in a way which makes apparent the invariants:

$$S_{T,2}^{(2)} = -\frac{1}{2} \frac{[\dot{u}_1^2 - (2Ti\hat{u}_1 - \dot{u}_1)^2]}{4T} \frac{[\dot{u}_2^2 - (2Ti\hat{u}_2 - \dot{u}_2)^2]}{4T} A(u_1 - u_2), \tag{126}$$

$$+ \frac{(2Ti\hat{u}_1 - \dot{u}_1)\dot{u}_1^2 - (2Ti\hat{u}_1 - \dot{u}_1)^3}{8T} i\hat{u}_2 B_1(u_1 - u_2) + (1 \leftrightarrow 2) \tag{127}$$

$$- \frac{(2Ti\hat{u}_1 - \dot{u}_1)\dot{u}_1^2}{4T} i\hat{u}_2 \frac{C'(u_1 - u_2)}{2}$$
$$- \frac{(2Ti\hat{u}_1 - \dot{u}_1)\ddot{u}_1}{4T} i\hat{u}_2 C(u_1 - u_2) + (1 \leftrightarrow 2), \tag{128}$$

in a way such that unwanted terms (with no $\hat{u}$ field associated to a time) cancel explicitly, apart from the last terms where they combine to gives rise to a boundary term $\frac{1}{2}\partial_{t_1}[\dot{u}_1^2 C(u_1 - u_2)]$. Accordingly, the B has been splitted into $B(u) = B_1(u) + C'(u)/2$. The function $B_1(u)$ must be a total derivative (see above) and its variation yields a boundary term. However, the invariance of the part cubic in the field in the

B_1 term is exact, which can be traced to an exactly invariant term in the unaveraged dynamical action

$$\frac{(2Ti\hat{u}_{rt} - \dot{u}_{rt})\dot{u}_{rt}^2 - (2Ti\hat{u}_{rt} - \dot{u}_{rt})^3}{4T} g(u_{rt}, r). \tag{129}$$

Expanding all terms above one can write explicitly

$$S_{T,2}^{(2)} = S_{T=0,2}^{(2)} + \frac{T}{2}[(i\hat{u}_1)^2 i\hat{u}_2\dot{u}_2 + (i\hat{u}_2)^2 i\hat{u}_1\dot{u}_1]A(u_1 - u_2) - \frac{T^2}{2}(i\hat{u}_1)^2(i\hat{u}_2)^2 A(u_1 - u_2) \tag{130}$$

$$+ \frac{3}{2}T[(i\hat{u}_1)^2 i\hat{u}_2\dot{u}_1 + (i\hat{u}_2)^2 i\hat{u}_1\dot{u}_2]B_1(u_1 - u_2) - T^2[(i\hat{u}_1)^3 i\hat{u}_2 + i\hat{u}_1(i\hat{u}_2)^3]B_1(u_1 - u_2). \tag{131}$$

Note that C does not give any bulk contribution at nonzero temperature. The FDT constraint only requires some boundary noise term for C. This is because C alone, with $A = B_1 = 0$ corresponds to a conservative dynamics [41].

These considerations show that to the order studied the $T > 0$ dynamical action is fully specified by the $T = 0$ action and the FDT constraints. Thus we do not need to introduce at this stage any new operator associated to finite T.

Having established that we are working with an appropriate action (and hence have not neglected any pertinent coupling constants/functions), we turn briefly to the effects of additional thermal contractions upon the FRG equations. Up to this point, the only such contractions we have included are the "diffusion" terms ($\tilde{T}_l\tilde{\Delta}''$, etc.) in the FRG equations for each coupling function. It appears natural to neglect most effects of temperature since it is an irrelevant variable under the FRG, while clearly these diffusion terms are crucial, since they are necessary to stabilize the boundary layer. Within this treatment, the zero temperature rule of conservation of powers of frequency still holds. More generally, however, one can a priori perform thermal contractions that feed downward (i.e., reduce the number of time derivatives) in the frequency hierarchy, in particular by thermally contracting fields containing time derivatives. For some simple such contractions, a preliminary calculation shows that a cancellation in fact occurs amongst the different "partners" required by the FDT, eliminating the unwanted mixing. We do not have, however, a general argument for such a mechanism of cancellation. Due to the complications of such a more general analysis, we will, however, proceed assuming this is generally true. We comment briefly further on this in the conclusion.

D. Dynamical boundary layer analysis: terms associated with averaged relaxation time

Having established that to this order no new terms arise due to temperature we now attempt to study the structure of the thermal boundary layer in the operators studied so far. We consider the level $m = 1$ in some detail. We add the effect of temperature to lowest naive order which is to add to the right-hand side of Eq. (91) the term $\tilde{T}\tilde{G}''(u)$, originating from the simple tadpole contraction of the G vertex.

From examination of this equation we expect that $\tilde{G}(u)$ supports a thermal boundary layer form for $u \sim \tilde{T}_l/\varepsilon$

$$\tilde{G}(u) = \epsilon\tilde{\chi} g\left(\frac{\epsilon\tilde{\chi}u}{\tilde{T}_l}\right), \tag{132}$$

with $g(0) = 0$, $g(x)$ an analytic function at $x = 0$, odd and positive for $x > 0$. It should match the fixed point form outside the boundary layer. For $u \sim O(1)$ and $\tilde{T}_l \ll \varepsilon$ we expect $\tilde{\Delta}(u)$, $\tilde{G}(u) \ll \tilde{\Delta}''(0) \sim \tilde{G}'(0) \sim \varepsilon^2\tilde{\chi}^2/\tilde{T}_l$. Thus in the outer region only three out of the several terms involving $\tilde{\Delta}$ and $\tilde{G}$ are non-negligible. For now we neglect feeding from third cumulants functions S and H, we return to them below. The fixed point for $\overline{G}(u)$ outside the TBL is then trivially

$$\overline{G}^*(u) = \left(\frac{2\tilde{\Delta}''(0)}{\tilde{G}'(0)} - 1\right)\tilde{\Delta}'^*(u). \tag{133}$$

Since at small argument $\tilde{\Delta}'^*(u) = -\varepsilon\tilde{\chi}$ it follows that $g(x) \to g_{+\infty} = 1 - 2\tilde{\Delta}''(0)/\tilde{G}'(0)$ a constant, for large x. Using the TBL form to evaluate $\tilde{G}'(0)$ yields

$$g_{+\infty} = 1 + \frac{2}{g'(0)}. \tag{134}$$

To analyze the boundary layer equation, we use the form (70) for $\tilde{\Delta}$ and similar forms for the third cumulant functions (72) for $\tilde{S}$ and

$$\tilde{H}(u_1, u_2, u_3) = (\epsilon\tilde{\chi})^2 h(\tilde{u}_1, \tilde{u}_2, \tilde{u}_3), \tag{135}$$

$$\tilde{u} = \frac{\epsilon\tilde{\chi}u}{\tilde{T}_l}. \tag{136}$$

The TBL equation for $g(x)$ is then found to be

$$0 = 2f''g + 3g'f' + g'(0)(f' - g) + g''(1 + f) + 2f'[f''(0) + f''] \tag{137}$$

$$+ s_{110}(0,0,\tilde{u}) + \tfrac{1}{3}[h_{010}(\tilde{u},0,0) - 2h_{001}(0,\tilde{u},0) - h_{100}(0,\tilde{u},0)], \tag{138}$$

where $s(u_1, u_2, u_3) = s_d^{(3)}(u_1, u_2, u_3)$, all these terms being multiplied by $\varepsilon^3\chi/\tilde{T}_l$ while the terms originating from rescaling are proportional to ε. This form will thus hold at scales such that $\tilde{T}_l \ll \varepsilon^2$.

For given functions f, s, h this equation is an eigenvalue problem for determining $g'(0)$. This can be seen since for large x the linear problem has one exponentially growing solution, in addition to the one matching the outer solution which converges to a constant. Thus $g'(0)$ must be tuned to select that solution. We illustrate this behavior in the approximation of neglecting all third cumulant functions. Then we recall that f satisfies

$$f'^2 + f''(1+f) = 1, \tag{139}$$

whose solution is $f(x)=\sqrt{1+x^2}-1$. Since, as discussed earlier G is a total derivative, it is possible to integrate the boundary layer equation once, and defining $g=-f'+\gamma'$ one obtains

$$(1+f)\gamma'' + 2f'\gamma' - g'(0)\gamma + (2f-1)[1+g'(0)] = 0,$$

with $\gamma(0)=\gamma'(0)=0$. One interesting solution but unrelated to the physics of interest here is $\gamma=0$, $g'(0)=-1$, i.e., $g=-f'$. It corresponds to an integrable set of models with a single exponential relaxation, which exactly obey the full FRG equations to one loop, and is studied in Appendix. A shooting procedure gives a solution $g(x)$ satisfying the proper boundary conditions with $g'(0)=2.646\pm0.001$.

Thus we find that the growth of $\overline{\eta_l}$ is determined by

$$\partial_l \overline{\eta_l} = \frac{\epsilon^2\tilde{\chi}^2}{\tilde{T}_l}[f''(0)+g'(0)]\overline{\eta_l} \tag{140}$$

yielding

$$\overline{\eta_l} \sim \exp\left(\alpha(1)\frac{\tilde{\beta}e^{\theta l}}{\theta}\right). \tag{141}$$

Clearly $\alpha(1)=3.646$ is a nontrivial number.

E. Terms associated with second moment of relaxation time

We now turn to the consideration of terms with $m=2$. As emphasized in Sec. II the principal quantities of interest are the cumulants of the friction, the second one being embodied in $A(u)$. The quantities B_1 and C also appear at this order, complicating the analysis. Since these embody somewhat different physics we will focus initially on $A(u)$ which fortunately satisfies equation (97) which is independent of B_1 and C.

We add the effect of temperature to lowest naive order which is to add to the right-hand side of Eqs. (97)–(99) the terms $\tilde{T}\tilde{A}''(u)$, $\tilde{T}\tilde{B}_1''(u)$, $\tilde{T}\tilde{C}''(u)$, respectively. These originate from the simple tadpole contractions.

1. Second moment of relaxation time $A(u)$

In Sec. III we pointed out the rapid divergence of the moments of the friction (relaxation time) $\overline{\eta}, \eta^{(2)}=A(0),\ldots$ driven by the low-temperature divergence of $\tilde{\Delta}''(0)$. In doing so we neglected all functional dependence [such as $A(u)$]. In the previous subsection we reconsidered the growth of the average friction $\overline{\eta}$, which clearly does not itself has any functional dependence. Instead the deviations of its growth from the prediction of Sec. III arise from a secondary mechanism of the feedback of $\tilde{G}'(0)$ into $\overline{\eta}$. Physically it corresponds to the cross correlation G of the friction $\eta(u,r)$ with the random force $f(u,r)$ producing an increased growth of $\overline{\eta}$.

We would now like to reconsider the growth of the second moment $A(0)=\eta^{(2)}$ including functional dependence. In this case already the leading effect of enhancement due to the divergence of $\tilde{\Delta}''(0)$ is nontrivial. Thus we will focus on it here primarily ignoring secondary effects of cross correlations between the random friction coefficient and the random force. In general these cross correlation effects enter the flow of A through G, H, and W defined in Eq. (81). Terms involving H and W have already been dropped in Eq. (97) for A. We will initially keep terms involving G in Eq. (97) but will drop them at a later stage of the analysis. It is not clear at this stage whether keeping these terms without simultaneously including the ones due to H and W would be consistent.

We then note that $\tilde{A}(u)$ satisfies a closed equation once $\tilde{G}(u)$ is known. We first consider the nature of its solution for $u\sim1$ outside the TBL. Doing so one notes the presence of several large terms proportional to $\tilde{\Delta}''(0)$, $\tilde{G}'(0)$. Balancing these large terms, we obtain the solution outside the TBL:

$$A_l(u) \sim \overline{\eta_l}^2 g_\infty \tilde{G}^{*\prime} = -\overline{\eta_l}^2 g_\infty^2 \tilde{\Delta}^{*\prime\prime}, \tag{142}$$

where g_∞ was defined above. Note that, as was the case for $\tilde{G}$ the convergence is very rapid due to the homogeneous part $\partial A=-2\tilde{G}'(0)A$. The important feature of this result is that $A(u)$ is of order $\overline{\eta_l}^2$ outside the TBL.

We are going to search for a TBL solution for A which grows faster:

$$\tilde{A}(u) = \overline{\eta_l}^\lambda \frac{\epsilon^2\tilde{\chi}^2}{T} h\left(\frac{\epsilon\tilde{\chi}u}{\tilde{T}_l}\right), \tag{143}$$

with $\lambda>2$. In order to match the above solution outside the TBL one should have $h(\infty)=0$. The TBL equation for h then reads

$$\{-\lambda[f''(0)+g'(0)] + 2f''(0) + 4f''\}h + (1+f)h'' + 4f'h' = 0. \tag{144}$$

Due to the presumed faster growth of $\tilde{A}$ than $\overline{\eta_l}^2$ $(\lambda>2)$ the feeding terms in Eq. (97) are negligible and have been dropped. As discussed above, since our principal interest is to compare the growth of the second relaxation moment parametrized by $A(0)$ relative to the growth of the mean $\overline{\eta}$, to be consistent we drop the analogous renormalization of $\overline{\eta}$ by $\tilde{G}'(0)$, i.e., set $g'(0)=0$ in Eq. (144). Numerical solution then yields

$$\lambda = 2.64. \tag{145}$$

The analysis is thus consistent since we find $\lambda>2$. To the order considered we therefore have

$$\overline{\eta^2}_l \sim \overline{\eta_l}^{2.64}\cdots \gg \overline{\eta_l}^2. \tag{146}$$

This gives, in the present approximation $\alpha(2)/\alpha(1)=2.64$. As seen in the previous subsection we expect both $\alpha(2)$ and $\alpha(1)$ to be both increased by inclusion of the effect of cross-correlations of friction and random force.

2. Growth of other $O(\omega^2)$ kinetic coefficients D, B, C

Due to the feedback of $A(0)$ into D we expect D to grow at least as fast as $\overline{\eta}_l^\lambda$. In the simplest scenario, indeed, all ω^2 quantities would scale the same in the same manner. However, we see no general reason why this need be the case. Indeed, examination of B_1 and C using the same truncation scheme as for A, shows that they grow faster. We sketch this analysis here. Consider first $\tilde{C}(u)$, which also feeds into the inertial mass D. We assume $\overline{C} \sim \overline{\eta}^\mu$, with $\mu > \lambda > 2$. With such growth of $\tilde{C}$, the feedback of $\tilde{C}'(0)$ into $\tilde{D}$ will overwhelm all other feeding terms, and we expect $\tilde{D} = \overline{\eta}^\mu \overline{D}$, with $\overline{D}$ scale independent. It is then natural to define $\tilde{C}(u) = -\tilde{D}\overline{C}(u)$. Equation (96) becomes

$$\partial_l \overline{D} = \{-2 + \mu[\tilde{G}'(0) - \tilde{\Delta}''(0)] - \overline{C}'(0)\}\overline{D}. \qquad (147)$$

Thus to leading order in $1/\tilde{T}_l$, one needs

$$\overline{C}(0) = \mu[\tilde{G}'(0) - \tilde{\Delta}''(0)]. \qquad (148)$$

Using the above forms, $\overline{C}$ satisfies

$$\begin{aligned}\partial \overline{C} = \{&2 - d + \zeta\partial_u - \tilde{\Delta}''(0) + \mu[\tilde{\Delta}''(0) - \tilde{G}'(0)] - 2\tilde{\Delta}''\}\overline{C} \\ &+ \tilde{T}_l \overline{C}'' - \tilde{\Delta}'\overline{C}'(0) - 3\tilde{\Delta}'\overline{C}' + [\tilde{\Delta}(0) - \tilde{\Delta}]\overline{C}'' - 2\tilde{\Delta}'\tilde{\Delta}''(0) \\ &- 2\tilde{\Delta}'\tilde{\Delta}'', \end{aligned} \qquad (149)$$

to leading order, i.e., dropping terms $\sim \overline{\eta}^{2-\mu}, \overline{\eta}^{\lambda-\mu}$, and neglecting feedback from higher cumulants as before. As for G and A, the outer solution for $u \sim O(1)$ is readily found equating the large terms $\sim \tilde{\Delta}''(0) + \overline{C}'(0) \sim 1/\tilde{T}_l$:

$$\overline{C} \sim \frac{\tilde{C}'(0) + 2\tilde{\Delta}''(0)}{(\mu-1)\tilde{\Delta}''(0) - \mu\tilde{G}'(0)}\tilde{\Delta}'(u) \;\; u \sim O(1). \qquad (150)$$

As before, for small u we make a TBL ansatz,

$$\overline{C}(u) = \epsilon\tilde{\chi}c(\epsilon\tilde{\chi}u/\tilde{T}_l), \qquad (151)$$

which yields an equation very similar to Eq. (138) for $g(x)$:

$$\begin{aligned}(1+f)c'' + 3f'c' + \{f''(0) - \mu[f''(0) + g'(0)] + 2f''\}c \\ - f'[2f''(0) - c'(0) + 2f''] = 0. \end{aligned} \qquad (152)$$

We require, to match Eq. (150), that c goes at a constant at large argument, and $c(0) = 0$ since c is an odd function. Furthermore, from Eq. (148), we have $c'(0) = -\mu[f''(0) + g'(0)]$. This formulates an eigenvalue problem for μ. As above, to solve, we use the (approximate) form for $f(x)$ in Eq. (139) and, for consistency as before set $g'(0) = 0$. A shooting procedure gives $\mu = 3.377$, indeed greater than λ as required for consistency. In summary we find the growth of the kinetic coefficients

$$\tilde{C}(u) \sim \tilde{D} \sim \overline{\eta}_l^{3.377}. \qquad (153)$$

Finally, we discuss the growth of $\tilde{B}_1$. Since it is fed by $\tilde{A}(0)$, it must grow at least as fast as $\overline{\eta}^\lambda$, so all other feeding terms on the last line of Eq. (98) are certainly negligible. Remarkably, even in the presence of the thermal $\tilde{T}_l\tilde{B}_1''$ term an asymptotically (for large l) exact solution can be found. In particular, one finds that the homogeneous (in $\tilde{B}_1$) part of the $\tilde{B}_1$ equation has an exact eigenfunction which is just $\tilde{B}_1(u,l) = \tilde{B}_1(l)$, a constant independent of u. This turns out to be the most unstable eigenfunction, with eigenvalue $-d - 3\tilde{\Delta}''(0)$. The exponential growth of this unstable eigenfunction is faster than that of $\tilde{A}(0)$, and hence dominates the flow at large scales. Hence, writing this relative in terms of $\overline{\eta}_l$ [neglecting the $g'(0)$ renormalization of $\overline{\eta}$ as before], one finds

$$\tilde{B}_1(u) = \tilde{B}_1(0)\overline{\eta}_l^3. \qquad (154)$$

V. CONCLUSION

We have through a series of successively better approximations arrived at a description of the growth of the moments of relaxation times (friction coefficient) encoded as eigenvalues of functional FRG equations. This final stage of analysis was carried out only for the mean and variance—the extension to higher moments is a formidable technical challenge. Nevertheless, already at this level we have observed how these functional eigenvalue problems provide a mechanism for describing a broad but nontrivial (i.e., not log normal) distribution of time scales. This is at variance with numerous other existing examples of systems exhibiting simpler log-normal tails which can be obtained from simpler nonlinear sigma model diagrammatic calculations, such as in disordered conductors [51]. A similar log-normal tail was indeed obtained in Sec. III from an approximate truncation of the FRG equation. A rather strong physical difference from the aforementioned quantum diffusion problem is the rapid exponential scale dependence of the relaxation times for $\theta > 0$, very different from the logarithmic dependence of two-dimensional weak localization corrections. It is an open question whether some less trivial distribution might arise at the metal-insulator transition in $d > 2$ and whether similar functional renormalization ideas might be useful in this context.

Many outstanding issues and extensions remain. Of these, the most fundamental are germane to both the dynamics and the statics [23,24]. In particular the very basic problem of perturbative control of the theory (most interestingly in the ϵ expansion) remains unsolved. This question, and the associated matching problem of relating, e.g., random force quantities such as the f_k in Eq. (72) defined deep within the boundary layer at $u=0$ to the zero-temperature ones occurring far outside for $u \sim O(1)$ are better addressed in the simpler context of the statics. We will refrain from commenting further upon them here.

Of the problems specific to the dynamics, perhaps most important is a systematic treatment of all thermal terms in the FRG. We have begun this program by classifying all operators associated with "thermal noises" in the effective action (Sec. IV C 1) consistent with the FDT. However, up to this point we have included the effects of nonzero temperature only through the leading "diffusion" terms ($\tilde{T}_l\tilde{\Delta}''$, etc.) in the FRG equations for each coupling function. As discussed in Sec. IV C 1, while this assumption is natural, we do not at this stage have a general justification for it. The importance of additional thermal contractions thus remains an important issue for further investigation.

Once these basic remaining issues in the FRG formulation are resolved, the present methods offer the opportunity to explore numerous physical problems. Obviously, equilibrium response and correlation functions are of considerable interest. Perhaps the approximate techniques of Sec. III (and Appendix E) may have an extension to the full functional description. It will also be valuable to reinvestigate the response to a uniform applied force in the creep regime [22] in light of the full dynamical structure of the thermal boundary layer exposed here. Applications of these ideas to nonequilibrium response and aging is also tantalizing. Similar approaches should be applicable to quantum problems in the Keldysh formalism. These and other applications of the present formulation certainly provide a broad scope for future progress in understanding glassy dynamics.

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APPENDIX A: SINGLE TIME SCALE CALCULATIONS—EQUILIBRIUM

1. Analytical results for the equilibrium response function

It is interesting to observe how the two *putative* scaling regimes described in Sec. II arise in a detailed calculation of the mean response function. To do so, we develop an FRG scheme to calculate directly the response function at arbitrary ω, k within the scaling regimes described above. It is necessary to follow the flow of the full wave vector and frequency dependence of the "kinetic" part of the MSR action. We therefore generalize the form in Eq. (11) to

$$S_0^l[u,\hat{u}] = \int_{r,r',tt'} i\hat{u}_{rt}(R_l^{-1})_{rt,r't'}u_{r't'} - \eta T\int_{r,t}(i\hat{u}_{rt})(i\hat{u}_{rt}), \tag{A1}$$

where, in a slight abuse of notation, we have denoted the quadratic MSR kernel by R^{-1}. Using Eq. (A1), we extend the FRG analysis leading to Eq. (15) to derive an RG equation for the response function. As before, the strategy is to integrate out spatial Fourier modes $\Lambda > k > \Lambda e^{-l}$, but now keeping the explicit time dependence. At this stage, we will *not* assume time-translational invariance, though we will specialize to this at a later stage. The FRG equation for R_l^{-1} is

$$\partial_l R_{q,l}^{-1}(t,t') = -\Gamma_l\Lambda_l^4\left(R_{\Lambda_l,l}(t,t') - \delta_{t,t'}\int_{t_i}^{t} dt'' R_{\Lambda_l,l}(t,t'')\right), \tag{A2}$$

where t_i is an initial time at which the system is prepared in some as yet unspecified state (or distribution of states). Equation (A2) is obtained formally by computing the correction to the (inverse) response function upon integrating out the modes in the shell, and using definitions (15) and (21). We perform this integration perturbatively in Δ (to first order), which gives the lowest order term in ϵ.

At the end we want the true response function $R_q(t,t')$. It will be obtained by integrating the flow from $l=0$ with the initial condition

$$R_{q,l=0}(t,t') = e^{-q^2(t-t')}\theta(t-t'), \tag{A3}$$

setting $\overline{\eta_0}=1$ for convenience, up to the scale l^* such that $\Lambda e^{-l^*}=q'$:

$$R_q(t,t') = R_{q,l=\ln(\Lambda/q)}(t,t'). \tag{A4}$$

This amounts to neglect contributions coming from the modes $k<q$, as is usually done in the RG. These are examined below.

Although the initial condition in Eq. (A3) is time-translationally invariant (TTI), the solution of the RG equation does not in general remain so, due to the presence of the initial time t_i. This leads to the aging properties to be discussed in Appendix B. The TTI regime is recovered in the limit $t_i\to\infty$ (for large but fixed finite size system) where one can set $R_k(t,t')=R_k(t-t')$. Then Eq. (A2) can be Fourier transformed in $t-t'$, $R_k(i\omega)=\int_{t>0}R_k(t)e^{-i\omega t}$ (i.e., Laplace transformed with $s=i\omega$) to obtain

$$R_k^{-1}(i\omega) = i\omega + k^2 + \Sigma_k(i\omega), \tag{A5}$$

$$\partial_k\Sigma_k(i\omega) = \tilde{\beta}k^{3-\theta}\left(\frac{1}{i\omega + k^2 + \Sigma_k(i\omega)} - \frac{1}{k^2+\Sigma_k(0)}\right), \tag{A6}$$

where we have defined a "self-energy" $\Sigma_k(i\omega)$ with initial condition $\Sigma_{k=1}(i\omega)=0$. To obtain Eq. (A6) one writes $\Sigma_k(i\omega)=\int_0^{\ln(\Lambda/k)}\partial_l R_{k,l}^{-1}(i\omega)$, uses the Fourier tranform of Eq. (A2) and differentiate with respect to k (we set from now on $\Lambda=1$). One can check that consistently $\Sigma_k(0)=0$, as requested by the statistical tilt symmetry, which we use from now on. Apart from special cases [43], Eq. (A6) does not admit analytical solution and we now analyze the various regimes of interest.

From Eq. (A6) one first finds the small ω behavior of $R_k^{-1}(i\omega)$ as

$$R_k^{-1}(i\omega) = k^2 + i\omega\eta_k + O(\omega^2), \tag{A7}$$

where η_k satisfies $\partial_k\eta_k=-\tilde{\beta}k^{-1-\theta}\eta_k$ which yields $\eta_k=k^2\tau_k$, i.e., one recovers as expected the single characteristic time scale τ_k given by Eq. (25).

To analyze further the higher order terms in $i\omega$ from Eq. (A6), we first consider the scaling regime $i\omega$, $k^2 \ll 1$ with $y = i\omega\tau_k$ fixed (which implies $i\omega \ll k^2$). Making the scaling ansatz

$$\Sigma_k(i\omega) = k^2 g(y), \tag{A8}$$

in Eq. (A6) gives the closed differential equation $yg' = g/(1+g)$, which has the implicit solution [taking into account the behavior of Σ for $i\omega \to 0$ from Eq. (A6)]

$$ge^g = y. \tag{A9}$$

Equations (A8), (A9) correspond to the Y scaling limit of Sec. II.

Equation (A8) is valid for finite y. As $y \to \infty$, we enter the logarithmic (X scaling limit) scaling regime, in which the scaling variable $x = k[\ln(1/i\omega)/\tilde{\beta}]^{1/\theta}$ is fixed and $i\omega$, $k^2 \ll 1$. Since $g(y) \to \infty$ in this limit, the first term on the right-hand side of Eq. (A6) can be neglected, leading to the ansatz

$$\Sigma_k(i\omega) = \tilde{\beta} k^{2-\theta} f(x), \tag{A10}$$

with $(2-\theta)f + xf' = -1$ from Eq. (A6) which determines the form of the scaling function $f(x)$ as

$$f(x) = \frac{1}{2-\theta}\left[\left(\frac{x_c}{x}\right)^{2-\theta} - 1\right], \tag{A11}$$

and the constant $x_c = (1/\theta)^{1/\theta}$ is determined by matching to Eq. (A8). This regimes exists only for

$$k < x_c[\ln(1/i\omega)/\tilde{\beta}]^{-1/\theta}, \tag{A12}$$

and thus corresponds to the limit of small wave vectors at fixed ω, or to relaxation times $\tau \ll \tau_k$ ($\tau \sim e^{\tilde{\beta}(x/k)^{-\theta}}$ for $x < x_c$). When $x \to x_c^-$ one crosses over to the $\mathcal{Y}$ regime.

We can check that these results merge smoothly with the result directly obtained at the upper critical dimension $d=4$. There the equation for $\Sigma_k(i\omega)$ becomes

$$\partial_k \Sigma_k(i\omega) = \tilde{\beta}\frac{k}{[\ln(1/k)]^2}\left(\frac{1}{i\omega + k^2 + \Sigma_k(i\omega)} - \frac{1}{k^2}\right), \tag{A13}$$

which yields the same two scaling regimes, the first one with $\tau_k \sim k^{-2}\exp\{\tilde{\beta}/[2k^2(\ln(1/k))^2]\}$ and the same scaling function $g(y)$ (A9) and the second one reading

$$\Sigma(k) = \frac{\tilde{\beta}}{\ln(1/k)}\left(-1 + 2\frac{\ln(1/k)}{\ln[\ln(1/i\omega)/\tilde{\beta}]}\right). \tag{A14}$$

We now turn to the calculation of the response function in the time domain. Consider first the regime $\mathcal{Y} = t/\tau_k$ fixed and $t \to \infty$, $k \ll 1$. Inverse Laplace-Fourier transforming Eq. (A5) and using Eq. (24) gives the scaling form

$$R_k(t) = \frac{1}{k^2\tau_k}\mathcal{G}(\mathcal{Y}), \tag{A15}$$

with

$$\mathcal{G}(\mathcal{Y}) = \frac{1}{2\pi i}\int_{-i\infty+\gamma}^{i\infty+\gamma}\frac{e^{y\mathcal{Y}}}{1+g(y)}. \tag{A16}$$

While we have not performed a complete analysis of the integral in Eq. (A16), the large time behavior can be extracted [44]. For $\mathcal{Y} \gg 1$, the integral is dominated by the vicinity of the branch point on the real negative axis at $y = -1/e$, leading to

$$R_k(t) \sim \frac{1}{k^2\tau_k}\left(\frac{t}{\tau_k}\right)^{1/2}\exp\left[-\frac{t}{e\tau_k}\right], \quad t \gg \tau_k. \tag{A17}$$

In the logarithmic scaling regime, we cannot simply invert the Fourier space result in Eq. (A10), as it does not extend over the entire frequency interval. Instead, we return to the defining RG equation for $R_k^{-1}(t)$, Eq. (A2). By inverting this formal equation, and again integrating down to scale k, we obtain an equation for $R_k(t)$ directly:

$$\partial_k R_k(t) = -2kR_k *_t R_k - \tilde{\beta}k^{3-\theta}\Bigg(R_k *_t R_k *_t R_k - R_k *_t R_k \int_{t'>0} R_k(t')\Bigg), \tag{A18}$$

where $*_\tau$ denotes a convolution. Note that, aside from the momentum dependence of the prefactors and the absence of derivative terms, Eq. (A18) bears a formal similarity to the mode-coupling equations of mean field models. In order to match the scaling expected from the above logarithmic frequency regime, we make the ansatz

$$R_k(t) \sim \frac{1}{t(\ln t)^{2-2/\theta}\tilde{\beta}^{2/\theta}}F\left[\mathcal{X} = \frac{1}{\tilde{\beta}}k^\theta \ln t\right], \tag{A19}$$

with $F[0]$ a constant. Inserting this in Eq. (A18), it is permissible to drop the first two terms in the logarthmic scaling regime, and moreover to approximate $\int_0^t dt' R(t-t')R(t') \approx 2R(t)\int_1^t dt' R(t')$. This yields

$$\theta\mathcal{X}F'[\mathcal{X}] = 2F[\mathcal{X}]\int_0^{\mathcal{X}} dz z^{-(2-2/\theta)}F[z]. \tag{A20}$$

The solution is

$$\int_{\ln(\mathcal{X}^{(2-\theta)/\theta}F[\mathcal{X}])}^{+\infty}\frac{dh}{\sqrt{(2-\theta)^2 + 4\theta e^h}} = -\left(\frac{1}{\theta}\right)\ln(\mathcal{X}/\mathcal{X}^*), \tag{A21}$$

where $\mathcal{X}^* = 1/\theta$ is the boundary of the regime, at which $F(\mathcal{X})$ diverges, signaling the onset of a regime of faster relaxation onto the regime $\mathcal{Y}$.

Having computed the response function $R_k(t)$ in the equilibrium TTI regime, we also obtain the time dependence of the connected correlation defined as $C_k(t) = \langle u_k(t)u_{-k}(0)\rangle - \langle u_k(t)\rangle\langle u_{-k}(0)\rangle$, with $C_k(t \to +\infty) = 0$ and $C_k(t=0)$ the equilibrium connected correlation. Indeed they are simply related through the fluctuation dissipation relation $\partial_t C_k(t) = -TR_k(t)$ or, in frequency space,

$$C_k(i\omega) = \frac{-T}{i\omega}[R_k(i\omega) - R_k(-i\omega)],. \tag{A22}$$

2. Discussion

We now pause and comment on the results of the FRG calculations we have just obtained. Let us first mention the nice features before stressing the unsatisfactory points below.

First we note that the Wilson scheme used directly on the response function within the single time scale assumption indeed yields, as we anticipated from general arguments in Sec. II, two distinct scaling regimes, the $\mathcal{Y}=t/\tau_k$ regime and the $\mathcal{X}=k^\theta \ln t$ regime, with scaling *forms* in Eqs. (A8), (A10), (A15), (A19). The scaling *functions* themselves were found to be nontrivial, with interesting analytical structure. While the existence of the $\mathcal{Y}=t/\tau_k$ regime seems to be a straightforward consequence of the assumption of a single time scale τ_k, the emergence of the $\mathcal{X}\sim k^\theta \ln t$ regime within this hypothesis is less obvious. Within the Wilson scheme, it seems to result from the system keeping a memory of a whole spectrum of smaller relaxation times $\tau\sim(\tau_k)^{x/x_c}$, $x<x_c$, generated during the coarse graining procedure and naturally appears here (while one would naively expect the largest one only, τ_k to play a role). It does have the form of *activated dynamics* since the scaling variable is truly $\mathcal{X}=(T/T^*)k^\theta \ln t$ and thus corresponds to crossing barriers of height$\sim k^{-\theta}$. That such an activated regime should exist is physically rather natural. Indeed we expect from simple droplet arguments that the equilibrium dynamics of mode k at large time difference (in general $t-t'$, denoted here t) is dominated by the rare active configurations with (at least) two quasidegenerate low free energy states at scale $L\sim 1/k$ of the system. The probability to find two nearly degenerate minima (on the scale of the thermal energy T) at scale $1/k$ is $\sim Tk^\theta$. One expects these two minima to be separated by a barrier U_b also scaling as $U_b\sim k^{-\theta}$ and thus when $T\ln t>U_b$ *equilibrium* thermally activated motion back and forth over this barrier [45] becomes active and gives rise to time-dependent correlations on the scale $\ln t\sim k^{-\theta}$. Our analytical result thus exhibits the correct scaling behavior and it is thus encouraging that such barrier crossing behaviour and scaling comes out of the present RG calculation.

Upon a closer look to our results in the X regime, everything works as if there is an effective distribution of smaller barriers $U_b=x'k^{-\theta}$ with a distribution of relaxation times $\tau=e^{\tilde{\beta}U_b}$ for $0<x'<x_c$. The total weight of this distribution being only $\sim Tk^\theta$ it can be written as $k^\theta/\tilde{\beta}\phi(x')$. $\phi(x')$ diverges at $x'=x_c$. This is easily seen, e.g., on the form for the correlations. Indeed, using the above FDT relation one obtains

$$C_k(t_1) - C_k(t_2) = \frac{T}{k^2}\frac{k^\theta}{\tilde{\beta}}\int_{k^\theta \ln t_1/\tilde{\beta}}^{k^\theta \ln t_2/\tilde{\beta}} \frac{du}{u^{2-2/\theta}}F(u), \tag{A23}$$

for the correlations in the logarithmic regime. In this expression, the T/k^2 equilibrium correlation is usually explained as $T/k^2=(1/k^{d+2\zeta})(Tk^\theta)$, i.e., the product of the size of a positional fluctuation between two degenerate states at scale k and the probability of this active configuration to occur. Thus we see that there is here an additional reduction by an extra factor Tk^θ, the total weight of barriers much smaller than τ_k (note that the above correlation variations within regime X are subdominant compared to the ones in regime Y, which really account for all but a small fraction of the total variation). Similarly one sees that the response corresponding to a barrier $x'k^{-\theta}$ can be written as

$$\frac{1}{k^2}\frac{1}{1+i\omega e^{\tilde{\beta}x'k^{-\theta}}} = \frac{1}{k^2}\frac{1}{1+e^{\tilde{\beta}k^{-\theta}(x'-x)}} \to \frac{1}{k^2}\theta(x-x'), \tag{A24}$$

as $k\to 0$ with $x=k^\theta \ln(1/i\omega)/\tilde{\beta}$. Thus averaging with the weight $(1/\tilde{\beta})k^\theta\phi(x')dx'$ yields exactly our result $R_k(i\omega)$ in regime X if one chooses

$$\int_0^x \phi(x')dx' = \frac{2-\theta}{\left(\frac{x_c}{x}\right)^{2-\theta}-1}. \tag{A25}$$

Another puzzling feature of the above results is the nonmonotonicity of the above scaling functions. As discussed below this is directly related to the assumption of a single time scale, and has prompted us to reconsider the whole calculation (at a high price of technical difficulty) in Sec. III. We see from Eq. (A6) that $\Sigma_k(i\omega)$ is a decreasing function of k always and that $R_k(i\omega)$ at fixed $i\omega$ is an increasing function of k for k small enough. Correspondingly, the real-time solution $R_k(t)$ is an increasing function of k at fixed t throughout the logarithmic regime and also in the short-time portion of the $t\sim\tau_k$ regime. Similarly, Eq. (A23) implies that the correlations are also increasing functions of k at fixed t. While this behavior is unexpected, we are presently unsure whether it is in fact unphysical. What is clear is that it is a consequence of the single time scale assumption. Indeed, the rather simple and apparently physical expression $R_k(t)=e^{-t/\tau_k}/(k^2\tau_k)$ is also increasing with k for small t/τ_k. It appears that one can argue fairly generally that, provided there exists a long-time regime with a *well-defined* τ_k, the response must be increasing with k for $t/\tau_k \lesssim 1$.

As discussed at length in the text, one does expect that the single time scale description is unsufficient and one should instead consider a distribution of time scales. Let us examine the question of monotonicity when $R_k(t)$ is simply a superposition of elementary relaxation processes. Discarding the subdominant k^{-2} prefactor and writing $\tau_k=e^{U_L}$ where $L=1/k$ is the scale, one can consider the average

$$R_k(t) = \int dU P_L[U] e^{-te^{-U}-U}. \tag{A26}$$

It is dominated by the saddle point $U^*(L,t)$ solution of

$$te^{-U} - 1 + \partial_U \ln P_L[U] = 0, \tag{A27}$$

and the condition for $R_k(t)$ to be an increasing function of L is

$$\partial_L P_L[U^*] > 0. \quad (A28)$$

For a Gaussian $\ln P_L[U] = -(U - L^\theta)^2/(2sL^\alpha)$ and $t=0$ (the worse point) one finds $U^* = L^\theta - sL^\alpha$ and $\ln P_L[U^*] = -L^\theta + (s/2)L^\alpha$. Thus one needs $\alpha \geqslant \theta$. Note, however, that this supposes that the Gaussian holds down to $U^* < 0$, which may not be the case in general. On the other hand, the only real condition concerns the monotonicity of the scaling function itself. Thus, one way to reduce the effect of nonmonotonicity is to increase the width of the distribution of time scales.

To close this discussion, it is useful to contrast the present situation of an elastic system with fast growing barriers with what happens in the marginal case $\theta=0$. This is realized for a periodic model in $d=2$, e.g., for the line of fixed points of the Cardy Ostlund model. There of course one expect a single scaling regime compatible with simple matching arguments. Setting $\theta=0$ in Eq. (A6) one finds the exact solution

$$k^2 = \left(1 + \frac{\Sigma_k(i\omega)}{i\omega}\right)^{-2/\tilde{\beta}} \left(1 + i\omega \frac{2}{2+\tilde{\beta}}\right) \quad (A29)$$

$$- \frac{2}{2+\tilde{\beta}}[i\omega + \Sigma_k(i\omega)], \quad (A30)$$

obtained writing $-d(k^2)/d\Sigma = (2/\tilde{\beta})[1 + k^2(i\omega + \Sigma)^{-1}]$. For $\omega \ll 1$ this yields the scaling form

$$\Sigma_k(i\omega) = k^2 g(y = i\omega k^{-z}), \quad z = 2 + \tilde{\beta},$$

$$y = g\left(1 + \frac{2}{2+\beta} g\right)^{\beta/2}, \quad (A31)$$

where z is the equilibrium dynamical exponent [note that for $\beta \to +\infty$ one recovers Eq. (A9)]. The self-energy nicely interpolates between $\Sigma_k(i\omega) \sim i\omega k^{-\tilde{\beta}}$ at small $i\omega \ll k^z$ (as also obtained from considering the flow of the uniform $\overline{\eta_l} \sim e^{\tilde{\beta} l}$) and $\Sigma_k(i\omega) \sim (z/2)^{\tilde{\beta}/z}(i\omega)^{2/z}$ at large $i\omega \gg k^z$. From there one obtains the response function $R_k(t) = k^{\tilde{\beta}}\mathcal{G}(tk^{-z})$, which is found to decay as in Eq. (A17) with $\tau_k \sim k^{-z}$ and a characteristic time $(1 + 2/\tilde{\beta})^{\tilde{\beta}/2}\tau_k$ (instead of $e\tau_k$ obtained for $\tilde{\beta} \to +\infty$) and which behaves as

$$R_k(t) \sim t^{-\tilde{\beta}/z}, \quad (A32)$$

in the limit $1 \ll t \ll k^{-z}$. The function $\mathcal{G}$ obeys the equation

$$\tilde{\beta}\mathcal{G} + (2+\tilde{\beta})\mathcal{Y}\mathcal{G}' = (-2+\tilde{\beta})\mathcal{G} *_{\mathcal{Y}} \mathcal{G} - \tilde{\beta}\mathcal{G} *_{\mathcal{Y}} \mathcal{G} *_{\mathcal{Y}} \mathcal{G}. \quad (A33)$$

Note that such as scaling function $\mathcal{G}$ of $\mathcal{Y} = tk^z$ leads to a trivial scaling regime in $\mathcal{X} = \ln t/\ln(1/k)$ reduced to a delta function at $\mathcal{X} = -z$. Note finally that even in this case, the scaling regime $R_k(t)$ is again nonmonotonous: it vanishes at $k=0$, increases up to k^*, with $t(k^*)^z = z\tilde{\beta}$ and decreases beyond.

APPENDIX B: SINGLE TIME SCALE CALCULATIONS—NONEQUILIBRIUM AND AGING

1. Response function and various regimes

The RG recursion relation for the response function derived above was not restricted to equilibrium, and it is thus interesting to write down the corresponding equations (for response and correlations) in the full nonequilibrium regime. Within the intrinsic limitations of the single time scale approach, this allows in principle to acces the aging properties of the system.

To obtain closed equations for the two time response function $R_k(t,t')$ once again iterates Eq. (A2) from the same (TTI) initial condition $R_{k,l=0}(t,t') = \theta(t-t')e^{-k^2(t-t')}$ up to $l = \ln(1/q)$, keeping t_i finite and making no TTI assumption. Thus the response satisfies the differential equation

$$R_k^{-1}(t,t') = \delta_{tt'}(\partial_{t'} + k^2) + \Sigma_k(t,t'), \quad (B1)$$

$$\partial_k \Sigma_k(t,t') = \tilde{\beta} k^{3-\theta}\left(R_k(t,t') - \delta_{tt'}\int_{t_i}^{t} dt'' R_k(t,t'')\right),$$

where matrix multiplication and inversion is with respect to (t,t'). It is more convenient to avoid the two time self-energy and write a closed equation for $R_k(t,t')$ as

$$\partial_k R_k(t,t') = -2k(R_k R_k)(t,t') \quad (B2)$$

$$-\tilde{\beta} k^{3-\theta}\left((R_k R_k R_k)(t,t') - \int_{t'}^{t} dt_1 R_k(t,t_1) R_k(t_1,t') \int_{t_i}^{t_1} dt'' R_k(t_1,t'')\right), \quad (B3)$$

with initial condition $R_{k=1}(t,t') = \theta(t-t')e^{-k^2(t-t')}$. The full analysis of this equation is quite complicated and we have not attempted it. We will give only a few features, at a naive level, which remain to be confirmed by a more detailed analysis left for the future.

The function $R_k(t,t')$ depends on three variables but in the limit $k \ll 1$, $t'-t_i \gg 1$, $t-t' \gg 1$ we expect that it takes scaling forms depending only on two variables. What these variables really are depends on the time regime, and one can identify several possible time regimes and subregimes. They can be classified as follows, where we indicate the form expected for the response function $R_k(t,t')$, by order of increasing time and time differences

$$\text{(I)} \quad \frac{\ln t'}{\ln \tau_k} < 1,$$

$$\text{(Ia)} \quad \frac{\ln(t-t')}{\ln t'} < 1: \quad \frac{g\left[k^\theta \ln(t-t'), \dfrac{\ln(t-t')}{\ln t'}\right]}{(t-t')\ln^\gamma(t-t')},$$

$$\text{(Ib)}\ \ t-t' \sim t':\ \frac{h\left[k^{\theta}\ln(t-t'),\frac{t-t'}{t'}\right]}{(t-t')\ln^{\delta}(t-t')},$$

$$\text{(Ic)}\ \ \frac{\ln t}{\ln t'}>1,\frac{\ln t}{\ln \tau_k}<1:\ \frac{f\left(k^{\theta}\ln t,\frac{\ln t}{\ln t'}\right)}{t'\ \ln^{\alpha} t'},$$

$$\text{(Id)}\ \ t\sim \tau_k:\ \frac{m\left(k^{\theta}\ln t',\frac{t}{\tau_k}\right)}{t'\ln^{\psi} t'}, \tag{B4}$$

$$\text{(II)}\ \ t' \sim \tau_k,$$

$$\text{(IIa)}\ \ \frac{\ln(t-t')}{\ln \tau_k}<1:\ \frac{F\left[k^{\theta}\ln(t-t'),\frac{t'}{\tau_k}\right]}{(t-t')\ln^{b}(t-t')},$$

$$\text{(IIb)}\ \ t-t' \sim \tau_k:\ \frac{G\left(\frac{t-t'}{\tau_k},\frac{t-t'}{t'}\right)}{k^2\tau_k}, \tag{B5}$$

$$\text{(III)}\ \ \frac{\ln t'}{\ln \tau_k}>1\ \ \text{equilibrium}\ \ R_k(t-t'),$$

$$\text{(IIIa)}\ \ \frac{\ln(t-t')}{\ln \tau_k}<1,\ \ \frac{F[k^{\theta}\ln(t-t')]}{(t-t')\ln^{2-2/\theta}(t-t')},$$

$$\text{(IIIb)}\ \ t-t' \sim \tau_k,\ \ \frac{G\left(\frac{t-t'}{\tau_k}\right)}{k^2\tau_k}.$$

Regime (III) is the equilibrium TTI regime, where the only dependence is in $t-t'$. There are two scaling forms possible corresponding to the two subregimes X (IIIa) and Y (IIIb) studied in Appendix A. Fully equilibrated regime (III) is expected here for very large times $t>t'\gg\tau_k$, and is somehow at variance with mean field models (where one always expect aging, e.g., for $t\sim t'$, even for very large t'). In regimes (I) and (II) the mode k at t' has not yet equilibrated, and the scaling functions ar now also function of $k^{\theta}\ln t'$ (regime I) or t'/τ_k (regime II), in addition of being functions of $t-t'$. In both regimes (I) and (II) if $t-t'$ is small one expects some kind of equilibrium regime. Indeed for $t-t'\sim O(1)$ we expect that there will be a fully TTI equilibrium regime, but it is also expected to be nonuniversal. A universal, *quasiequilibrium* regime is expected, however, for $t-t'\sim t'^{u}<t'$, $u<1$ [regimes (Ia) and (IIa)]. As the time difference increases it should crossover at $t-t'\sim t'$ to an intermediate aging regime [regimes (Ib) and (IIb)]. Regime I is most complex as there one expects two later regimes as $t-t'\sim t\sim t'^{v}$, $v>1$ crossing over to yet another scaling regime when t reaches τ_k. It is interesting to note that either in Sinai model [46] or even more clearly in the 1D random field Ising model [7] such regimes are also expected, some have been studied demonstrated and studied in detail (there is also an equilibration time scale analogous to τ_k).

In the determination of all the above regimes the quantity

$$\mu_k(t)=\int_{t_i}^{t}dt''R_k(t,t''), \tag{B6}$$

which appears in Eq. (B3) plays an important role. It is a function of t alone. It satisfies the equation

$$\partial_k\mu_k(t)=\int_{t_i}^{t}dt_1R_k(t,t_1)\left[-2k\mu_k(t_1)-\tilde{\beta}k^{3-\theta}\left(\mu_k(t_1)^2-\int_{t_i}^{t_1}dt'R_k(t_1,t')\mu_k(t')\right)\right]. \tag{B7}$$

Its value can be understood by using the STS covariance under $u_{rt}\to u_{rt}+v_r$, where v_r is an arbitrary function. In a general nonequilibrium situation, the STS gives constraints relating different initial conditions at $t=t_i$. It can be written as $\ln Z[h_{kt},\hat{h}_{kt},u_{k,t=0}=0]=Z[h_{kt}+k^2v_k,\hat{h}_{kt},u_{k,t=0}=v_k]-\int_{kt}\hat{h}_{kt}v_k$, where the initial condition is explicitly indicated. It thus immediately yields

$$\mu_k(t)=\frac{1}{q^2}\left(1-\frac{\delta\langle u_q(t)\rangle}{\delta u_q(t_i)}\right). \tag{B8}$$

When $t_1\gg\tau_k$ we expect that the influence of the initial condition on mode k has been washed out, and we find the equilibrium constraint $\lim_{t\to+\infty}\mu_k(t)=\int_0^{+\infty}R_k(\tau)d\tau=1/k^2$ [which, combined with FDT gives $C_k(0)-C_k(\infty)=T/k^2$]. Thus, we expect $\mu_k(t)$ to take the form

$$\mu_k(t)=k^{-2+\sigma}m(k^{\theta}\ln t)\ \ \frac{\ln t}{\ln \tau_k}<1, \tag{B9}$$

$$\mu_k(t)=k^{-2}\mathcal{M}(t/\tau_k)\ \ t\sim\tau_k, \tag{B10}$$

with $\mathcal{M}(\infty)=1$ and a reduction k^{σ} in the short time regime compared to asymptotic one, with an interpretation in terms of the susceptibility to initial condition being almost 1, presumably with some rare (droplets ?) configurations exhibiting decorrelation.

To discuss the specific choice of the scaling functions and prefactors we proceed as follows. Let us consider regime (I). We have found that with the forms of the prefactors in subregimes (Ic) and (Ia) indicated above we could obtain from Eq. (B7) nontrivial equations for the scaling functions. The regime (Ib) is then necessary to match (Ic) and (Ia). Next, with the forms conjectured for (Ia,b,c) the terms in Eq. (B7) scale, respectively, as 1, $k^{2-\theta(1-\alpha_i)}$, $k^{4-\theta-2\theta(1-\alpha_i)}$, $k^{2-\theta-\theta(1-\alpha_i)+\sigma}$ (where the first term is the derivative) with $\alpha_i=\gamma,\ \delta,\ \alpha$, respectively, in each subregime. Thus either $\sigma\leqslant\theta/2$ and $\theta(1-\alpha)=2-\theta+\sigma$ and only the last term counts or $\sigma\geqslant\theta/2$ and $\theta(1-\alpha)=2-\theta/2$ (in equilibrium regime III one had $\sigma=0$ leading to $\alpha=2-\theta/2$). Here, we see that if $\mu_k(t)$ is determined by an integration over regimes (Ia,b,c) as is natural, it implies $\sigma=2-\max_{\alpha_i=\gamma,\delta,\alpha}\theta(1-\alpha_i)$, and one sees that $\sigma=\theta/2$ and

$$\gamma=\delta=\alpha=\frac{3}{2}-\frac{2}{\theta}, \tag{B11}$$

is the only solution. Note that this contradicts the naive expectation that the form in the quasiequilibrium regime (Ia) would scale as the equilibrium form IIIa [they differ by a power of a $\ln(t-t')$]: to get $\gamma=2-2/\theta$ would require $\sigma=0$, and some argument that the value of $\mu_k(t)$ is controlled by $t-t''$ in the short time nonuniversal regime.

Accepting the above scenario as reasonable we find the equation for the scaling function $f(x,u)$ as

$$\begin{aligned}\theta x\partial_x f(x,u) = \tilde{\beta}x^{2(1-\alpha)} &\times\left[\int_u^1 \frac{du_1}{u_1^\alpha}f(x,u_1)f\left(xu_1,\frac{u}{u_1}\right)\right.\\ &\times\int_0^{u_1}\frac{du_2}{u_2^\alpha}f\left(xu_1,\frac{u_2}{u_1}\right)-\int_u^1\frac{du_1}{u_1^\alpha}\\ &\left.\times\int_u^{u_1}\frac{du_2}{u_2^\alpha}f(x,u_1)f\left(xu_1,\frac{u_2}{u_1}\right)f\left(xu_2,\frac{u}{u_2}\right)\right].\end{aligned} \tag{B12}$$

The 0 bound in the integral really comes from the ration $\ln t_i/\ln t$ assumed to be very small. The first term in the right-hand side of Eq. (B3) gives a subdominant contribution. Similar equations hold for the other regimes. We have not attempted to analyze further these equations at this stage. This would be necessary to fully confirm the self-consistency of the scenario proposed here.

2. Correlation function

Let us now indicate the RG equation obeyed by the correlation function. It is obtained from considering the full local quadratic term in the running effective action

$$-\frac{1}{2}\int_{r,t>t_i,t'>t_i}(i\hat{u}_{rt})(i\hat{u}_{rt'})U_l(t,t'). \tag{B13}$$

One can also decompose it as $U_l(t,t')=V_l(t,t')+\Delta_l(0)$ by extracting the persistent part (disorder), requiring that $\lim_{t,t',t-t'\to+\infty}V_l(t,t')=0$. To lowest order $O(\Delta)$ U_l is corrected and flows as follows:

$$\begin{aligned}\partial_l U_l(t,t') = -\Gamma_l\Lambda_l^4\Big[&\tfrac{1}{2}C_{\Lambda e^{-l},l}(t,t)+\tfrac{1}{2}C_{\Lambda e^{-l},l}(t',t')\\ &-C_{\Lambda e^{-l},l}(t,t')\Big],\end{aligned} \tag{B14}$$

where we assumed a flat initial condition $u_{rt=0}=0$ (otherwise it should be added) and to this order U_l remains local. The persistent part of Eq. (B14) yields $\partial_l\Delta_l(0)=-T\Gamma_l\Lambda_l^2$ in agreement with Eqs. (15), (17) [using that the persistent part of the parenthesis in Eq. (B14) is the equilibrium connected correlation $C^{eq,c}_{\Lambda e^{-l},l}=T_l\Lambda^{-2}e^{2l}$]. Substracting it yields the flow of V_l. One closes the equations determining C_l, U_l using $C_{k,l}(t,t')=[R_{k,l}\cdot U_l\cdot R_{k,l}](t,t')$. Equivalently one can separate the effect of the random force part of the disorder in the correlation and write $C_{k,l}(t,t')=\tilde{C}_{k,l}(t,t')+\Delta_l(0)\mu_{k,l}(t)\mu_{k,l}(t')$ [with $\mu_{k,l}(t)=\int_0^t dt' R_{k,l}(t,t')$] with two closed equations for $\tilde{C}_l$ and V_l using $\tilde{C}_{k,l}(t,t')=[R_{k,l}\cdot V_l\cdot R_{k,l}](t,t')$.

Proceeding as above, to determine the correlation one defines $U_k=U_{l=\ln(\Lambda/k)}$ (similarly for V_k) and obtains

$$\partial_k U_k(t,t')=\tilde{\beta}k^{3-\theta}\left[\tfrac{1}{2}C_k(t,t)+\tfrac{1}{2}C_k(t',t')-C_k(t,t')\right], \tag{B15}$$

with $C_{k,l=\ln(\Lambda/k)}(t,t')=C_k(t,t')$, which should be solved along with

$$C_k(t,t')=\int_{t_i}^t dt_1\int_{t_i}^{t'}dt_2 R_k(t,t_1)U_k(t_1,t_2)R_k(t',t_2), \tag{B16}$$

with initial conditions at $k=1$:

$$U_k(t,t')=2\eta T\delta_{tt'}+\Delta(0)\ (k=1), \tag{B17}$$

$$\begin{aligned}C_k(t,t')=&\frac{T}{k^2}\left(e^{-k^2|t-t'|}-e^{-k^2(t+t')}\right)\\ &+\frac{\Delta(0)}{k^4}(1-e^{-k^2t})(1-e^{-k^2t'})\ (k=1).\end{aligned} \tag{B18}$$

Using the equation for $\partial_k R_k$ one can also write the closed equation for $C_k(t,t')$ as

$$\begin{aligned}\partial_k C_k(t,t')=\int_{t_1,t_2>t_i}&[R_k(t,t_1)C_k(t_2,t')+R_k(t',t_1)C_k(t_2,t)]\\ &\times\left[-2k\delta_{t_1,t_2}-\tilde{\beta}k^{3-\theta}\left(R_k(t_1,t_2)-\delta_{t_1,t_2}\right.\right.\\ &\left.\left.\times\int_0^{t_1}dt''R_k(t_1,t'')\right)\right]+\tilde{\beta}k^{3-\theta}R_k(t,t_1)R_k(t',t_2)\\ &\times\left(\frac{1}{2}C_k(t_1,t_1)+\frac{1}{2}C_k(t_2,t_2)-C_k(t_1,t_2)\right),\end{aligned} \tag{B19}$$

and initial condition (B18). Alternatively, one can work with $\tilde{C}_k$ and V_k, which have a more complicated equation but simpler initial conditions

$$V_k(t,t')=2\eta T\delta_{tt'}\ (k=1), \tag{B20}$$

$$\tilde{C}_k(t,t')=\frac{T}{k^2}\left(e^{-k^2|t-t'|}-e^{-k^2(t+t')}\right)\ (k=1).$$

One easily checks that upon the assumption of time translational invariance as should hold in the equilibrium regime, the equation for $C_k(t,t')=C_k(t-t')$ becomes, as expected, equivalent to the one for $R_k(t-t')$ via the FDT relation. Further study of the nonequilibrium equations, including the determination of the FD violation ratio $X(t,t')$ in the various regimes is left for forthcoming publications.

APPENDIX C: CORRECTIONS TO THE *f*-TERM BY PINNING DISORDER

Let us give some details about the calculation of the graphs in Figs. 3, 4. The correction to the effective action to lowest order in T coming from the cross term $F\Delta$ reads

$$\delta\Gamma = \Big\langle \frac{1}{2}\int_{r,r_1,t,t'} (i\hat{u}_{rt} + i\delta\hat{u}_{rt})(i\hat{u}_{rt'} + i\delta\hat{u}_{rt'})\Delta\big[u_{rt} - u_{rt'} + \delta(u_{rt} - u_{rt'})\big] \times F\Big[z_{r_1} + \int_{t_1} i\delta\hat{u}_{r_1t_1}\partial_{t_1}u_{r_1t_1} + \int_{t_1} i\hat{u}_{r_1t_1}\partial_{t_1}\delta u_{r_1t_1} + \int_{t_1} i\delta\hat{u}_{r_1t_1}\partial_{t_1}\delta u_{r_1t_1}\Big]\Big\rangle^{1PI}_{\delta u,\delta\hat{u}}, \tag{C1}$$

with $z_r = \int_t i\hat{u}_{rt}\partial_t u_{rt}$ and the averages over δu, $\delta\hat{u}$ are restricted to one-particle irreducible graphs. This splits into contributions corresponding to graphs (a)–(c) in Fig. 3 which evaluate respectively as (dropping all terms which do not correct F):

$$\delta\Gamma^{(a)} = \frac{1}{2}\int_{r,r_1,t,t'} i\hat{u}_{rt}i\hat{u}_{rt'}\frac{1}{2}\Delta''(0)\Big\langle(\delta u_{rt} - \delta u_{rt'})^2 F \times\Big[z_{r_1} + \int_{t_1} i\delta\hat{u}_{r_1t_1}\partial_{t_1}u_{r_1t_1}\Big]\Big\rangle = -\frac{1}{2}\Delta''(0)R^2_{q,\omega=0}\int_r\Big(z_r^2F''[z_r] - \int_{tt'} i\hat{u}_{rt'}i\hat{u}_{rt}(\partial_t u_{rt})^2\Big), \tag{C2}$$

$$\times\,\delta\Gamma^{(b)} = \frac{1}{2}\int_{r,r_1,t,t'}\Delta(u_{rt} - u_{rt'})\Big\langle i\delta\hat{u}_{rt}i\delta\hat{u}_{rt'}F \times\Big[z_{r_1} + \int_{t_1} i\hat{u}_{r_1t_1}\partial_{t_1}\delta u_{r_1t_1}\Big]\Big\rangle = \frac{1}{2}R^2_{q,\omega=0}\int_{r,t,t'}\partial_t\partial_{t'}\big[\Delta(u_{rt} - u_{rt'})\big]i\hat{u}_{rt}i\hat{u}_{rt'}F''[z_r], \tag{C3}$$

$$\times\,\delta\Gamma^{(c)} = \frac{1}{2}2\int_{r,r_1,t,t'} i\hat{u}_{rt}\Big\langle i\delta\hat{u}_{rt'}\Delta(u_{rt} - u_{rt'} + \delta u_{rt} - \delta u_{rt'})F\Big[z_{r_1} + \int_{t_1}\delta i\hat{u}_{r_1t_1}\partial_{t_1}u_{r_1t_1} + \int_{t_1}\partial_{t_1}\delta u_{r_1t_1}i\hat{u}_{r_1t_1}\Big]\Big\rangle = R^2_{q,\omega=0}\int_{r,t,t'} i\hat{u}_{rt}i\hat{u}_{rt'}\partial_t u_{rt}\partial_{t'}\Delta'(u_{rt} - u_{rt'})F''[z_r], \tag{C4}$$

as well as the graph in Fig. 4:

$$\delta\Gamma^{(3)} = \frac{1}{2}2\int_{r,r_1,t,t'} i\hat{u}_{rt}\Big\langle i\delta\hat{u}_{rt'}\Delta(u_{rt} - u_{rt'} + \delta u_{rt} - \delta u_{rt'})F \times\Big[z_{r_1} + \int_{t_1}\delta i\hat{u}_{r_1t_1}\partial_{t_1}\delta u_{r_1t_1}\Big]\Big\rangle = \int_{r,t,t'} i\hat{u}_{rt}\partial_{t'}\Delta'(u_{rt} - u_{rt'})F'[z_r]\int_{t_1} R_{q,t-t_1}R_{q,t_1-t'}. \tag{C5}$$

This yields the result (48) in the text.

APPENDIX D: MAPPING OF RANDOM FRICTION MODEL ONTO POLYMER AND RELATED PROBLEMS

The random friction model (in its nontrivial $T=0$ limit) can be mapped formally onto various other problems, such as, after disorder averaging, the statistical mechanics of a pure self-interacting chain (e.g., a self-avoiding walk problem) or, prior to averaging, to some random diffusion models, e.g., depolarization of a spin diffusing in a random magnetic field. Concerning its behavior one should distinguish between the genuine model [with a fixed distribution $P(\eta)$] and the effective one which appear as a coarse grained version of the pinning problem, in which $P(\eta)$ flows and becomes very broad.

First setting $P(r,t) \sim \eta(r)u_{rt}$ one sees that Eq. (28) [with $f(r,u)=0$ and $T=0$] is the Fokker-Planck equation $\partial_t P = \nabla D(r)[\nabla + \nabla V(r)]P$ for the diffusion of a particle with a random diffusion coefficient $D(r)=1/\eta(r)$ in a random potential $V(r)=-\ln\eta(r)$, of equilibrium measure $P_{\rm eq}(r)=e^{-V(r)} = \eta(r)$. When $\eta(r)$ is uncorrelated from site to site one does not expect any anomalous behavior in any dimension, except if the distribution of η has broad tails (e.g., algebraic would yield anomalous power law diffusion). In the effective model $V(r)$ becomes Gaussian and grows with scale which corresponds to a particle localized in some regions of space.

A complementary picture can be developed based on a mapping onto a self-interacting chain. The response function $R_{rt,r't'} = d\delta u_{rt}/dh|_{h=0}$ of this model is obtained by solving

$$\big[\eta(r)\partial_t - \nabla^2\big]\delta u_{rt} = h\,\delta(r-r')\delta(t-t'), \tag{D1}$$

with initial condition $\delta u_{rt=0}=0$. This implies $\delta u_{rt}=0$ for all $t<t'$. Thus the response is a function of $t-t'$ alone and its Laplace-Fourier transform $s=i\omega$ in any given random environment, can be written as

$$R_{rr'}(s) = \Big\langle r\Big|\frac{1}{-\nabla^2 + s\,\eta(r)}\Big|r'\Big\rangle = \int_0^{+\infty} du\langle r|e^{-uH}|r'\rangle, \tag{D2}$$

where $H=-\nabla^2+s\,\eta(r)$, which has a positive spectrum for $s>-s*$. The value $s*$ at which H develops an eigenstate of zero eigenvalue (e.g., $s*=1/\eta_{\max}$ in the "classical" limit $c \to 0$) gives the large time decay of $R_{rr'}(t) \sim e^{-s*t}$.

We can also write, in the Fourier domain, using the Feynman Kac formula

$$R_{rr'}(i\omega) = \int_0^{+\infty} du e^{-(\mu + i\omega\overline{\eta})u} \int_{x(0)=r'}^{x(u)=r} Dx(v),$$

$$\exp\left\{ - \int_0^u dv \left[\frac{1}{4}\left(\frac{dx}{dv}\right)^2 + i\omega\delta\eta(x(v)) \right] \right\}, \qquad \text{(D3)}$$

with the "time" variables u and v. We have splitted $\eta(x) = \overline{\eta} + \delta\eta(x)$ for convenience and added a small mass term μ for convenience. In this form the problem has the form of a spin decoherence problem, the integral being dominated by paths which average well over the random relaxation times, rather than paths with multiple returns to the same region which average poorly and then cancel incoherently (details about the mapping and special distribution of noise can be found in Ref. [47]).

Averaging over disorder leads, for small disorder, to

$$R_{r-r',\omega} = \int_0^{+\infty} du e^{-(\mu + i\omega\overline{\eta})u} Z\left(r - r', u, g = \frac{1}{2}\omega^2 \eta_2\right),$$

$$Z(r - r', u, g) = \int_{x(0)=r'}^{x(u)=r} Dx(v), \qquad \text{(D4)}$$

$$\exp\left(- \int_0^u dv \frac{1}{4}\left(\frac{dx}{dv}\right)^2 - \int_0^u dv dv' g \delta(x(v) - x(v')) \right),$$

which is the partition function of a self-avoiding walk in the Edwards representation. We have retained only the second moment η_2 of $\delta\eta(x)$, but of course the full interaction could be written using $F[z]$. The theory is described by a nontrivial fixed point in $d<4$ [related to the $n=0$ $O(n)$ model with mass $(\mu + i\omega\overline{\eta})$ and coupling $\tilde{g} \sim g\Lambda^{4-d}$]. This is compatible with the previous conclusions for the F theories using perturbation theory if we take t as a simple index, with no power counting dimension (it plays a role somewhat similar to the replica index a). The correction to g by g^2 comes from the contraction of two η_2 vertices, of the form $\delta(\omega^2\eta_2) = \omega^4\eta_2^2$ and is indeed logarithmically divergent in $d=4$. Note that the mass term is thus relevant at the fixed point $\tilde{g} = g*$.

This analysis yields information at finite time. The Ginsburg criterion gives the critical regime as $r > g^{-1/(4-d)}$ or $u > g^{-2/(4-d)}$, in which $Z(r-r', u, g)$ takes the scaling form

$$Z(r,u,g) = u^{-\nu d} F[r u^{-\nu}] Z(q=0,u,g), \qquad \text{(D5)}$$

$$Z(q=0,u,g) \sim u^{\gamma-1} e^{-s_c(g)u}, \qquad \text{(D6)}$$

with $s_c(g) = cg$ for $d>2$, $s_c(g) = cg \ln(1/g)$ for $d=2$, and $s_c(g) = cg^{2/3}$ for $d=1$. Thus for $d>2$ we expect that

$$R_{q,\omega} \approx \frac{1}{i\overline{\eta}\omega + c\,\eta_2\omega^2 + q^2}, \qquad \text{(D7)}$$

in the noncritical regime, while we expect

$$R_{q=0,\omega} \approx \left(\frac{1}{i\overline{\eta}\omega + c\,\eta_2\omega^2}\right)^{\gamma}, \qquad \text{(D8)}$$

i.e., $R_{q=0,t} \sim t^{\gamma-1} e^{-c\eta_2 t/\overline{\eta}}$ in the critical regime, and for $q>0$, the appropriate scaling function of $rt^{-\nu}$.

The critical regime correspond to

$$\overline{\eta}\omega > \left(\frac{\overline{\eta}^2}{\eta_2}\right)^{2/d}, \qquad \text{(D9)}$$

which for the genuine model gives a singularity only at finite (true) time (see, however, Ref. [47] for possibly more radical effect of non-Gaussian disorder). However, in the limit of very broad disorder, as in the effective model, one has $\eta_2 \gg \overline{\eta}^2$ and thus the critical singularity moves to small ω.

Note that for $d<2$ the behavior is more radical as one expects, e.g., in $d=1$:

$$R_{q,\omega} \approx \frac{1}{i\overline{\eta}\omega + c(\eta_2)^{2/3}\omega^{4/3} + q^2}. \qquad \text{(D10)}$$

Thus to conclude, in the absence of pinning disorder at $T=0$ the F term is preserved but generate higher order time derivative terms. The theory can be rescaled so as to possess a nontrivial finite ω, finite disorder term, which presumably in $d>2$ produces only preexponential algebraic corrections to the leading behaviors given by the most relevant $\overline{\eta}$ term. For the effective model this critical behavior should be observable even at small ω.

APPENDIX E: FULL FLOW OF THE TRUNCATED EFFECTIVE ACTION

In this appendix we discuss an approach to the calculation of the mean response function extending the approximate FRG scheme of Sec. III, but still neglecting the functional dependence of operators considered in Sec. IV. In Sec. III the broad distribution of time scales was embodied in the so-called F term. We uncover here an interesting structure of additional operators in the spirit of the more complete set of moments $\overline{\langle t^{P_1}\rangle_L \cdots \langle t^{P_N}\rangle_L}$ discussed in the Introduction. Recall that in Sec. III B, we showed that, although the F term did not renormalize itself, it did generate higher derivative terms. Such terms *can* contribute to the frequency dependence of the response function. We studied in the previous appendix the response function of the pure random friction model, which does generate higher derivative terms, but neglects the scale dependence generated by the pinning disorder. Here we consider these two effects in tandem, hence modifying the results for $R_k(t)$, $R_k(\omega)$ within the purely random friction model. While we have not been able to obtain simple expressions in this more complete (albeit still nonfunctional) approximation, one can go quite far in reducing the problem to one of applied mathematics.

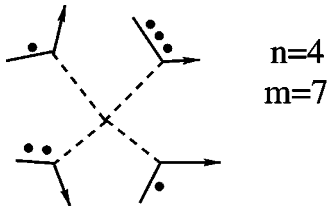

FIG. 12. Graphs with only $n=1$ vertices which enter in the response function (a) and its self-energy (b).

1. Generalized random friction model

To proceed further one needs to construct a more systematic approach where all possible important terms in the dynamical action functional are included. We will generalize the F term (at zero temperature for simplicity) in the form

$$S_{\text{kin}} = \sum_{n=1}^{\infty} \sum_{p_1\cdots p_n \geq 1} F^{(n)}_{p_1\cdots p_n} \int_{rt_1\cdots t_n} i\hat{u}_{rt_1} \cdots i\hat{u}_{rt_n} \times \partial_{t_1}^{p_1} u_{rt_1} \cdots \partial_{t_n}^{p_n} u_{rt_n}, \tag{E1}$$

and we will often denote by $m=p_1+\cdots+p_n$ the total number of time derivative in a given term of the sum. This form of the action neglects terms with products of time derivatives of u_{rt} at the same time, as well as statistically translationally invariant functional dependence, e.g., on $u_{rt_1}-u_{rt_2}$. However, it does include considerably more physics, and as we will see, enough generality to approach the problem of computing the averaged response functions. This kinetic part of the action corresponds to the following generalization of the random friction model:

$$\sum_{m=1}^{+\infty} \eta_m(r) \partial_t^m u_{rt} = \nabla^2 u_{rt} + F(u_{rt}, r), \tag{E2}$$

with $\overline{[\eta_{p_1}(r_1)\cdots \eta_{p_n}(r_n)]}_C = n!(-1)^{n+1} F^n_{p_1,\ldots,p_n} \delta_{r_1,\ldots,r_n}$.

The response function is related to the lowest $(n=1)$ member of this hierarchy via

$$R_k^{-1}(i\omega) = k^2 + \tilde{\Sigma}_k(i\omega), \tag{E3}$$

$$= \tilde{\Sigma}_k(i\omega) = \sum_{m=1}^{\infty} F^{(1)}_m (i\omega)^m, \tag{E4}$$

and within the Wilson scheme the true physical response function $R_k^{-1}(i\omega)$ is obtained via the same formula using the running $F^{(1)}_m|_{l=\ln(\Lambda/k)}$. This is represented graphically in Fig. 12.

One can carry perturbation theory using the generalized F. The vertices are shown in Figs. 13, 14. In this notation, there are a variety of important one loop diagrams to be considered. These are shown in Fig. 15. Schematically, these contributions give rise to an RG equation for the $F^{(n)}_m$ of the form

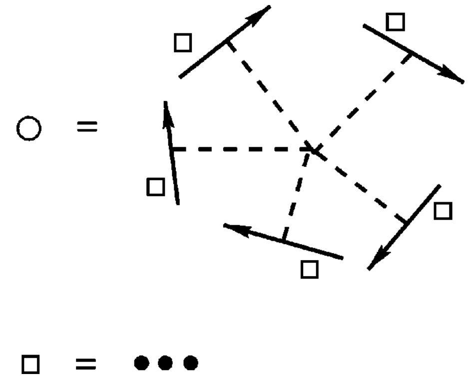

FIG. 13. Graphical representation of an F vertex $F^n_{p_1\cdots p_n}$ with $n=4$, $p_1=1$, $p_2=2$, $p_3=1$, $p_4=3$, $m=\Sigma_i p_i=7$. Dots represent the number of time derivatives (i.e., power of frequency factor), each leg has a different frequency.

$$\partial_l F^{(n)}_m = \Delta F^{(n)}_m + F^{(n+1)}_m + F^{(n')}_{m'} F^{(n+2-n')}_{m-m'} + \Delta F^{(n')}_{m'} F^{(n+1-n')}_{m-m'} + \Delta^2 F^{(n')}_{m'} F^{(n-n')}_{m-m'} + \cdots. \tag{E5}$$

In Eq. (E5), we have neglected coefficients, powers of l, and fine distinctions such as the precise form of Δ which appears in a given term. Repeated (primed) indices other than m and

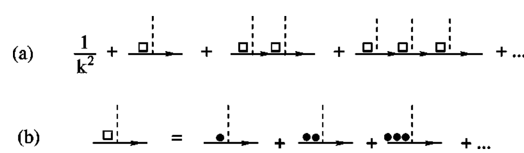

FIG. 14. Compact notation for a generic F vertex. The open circle represents an F vertex with an arbitrary number of legs n not shown. On incoming (u) lines, an arbitrary number of time derivatives (powers of ω) are indicated by an open square.

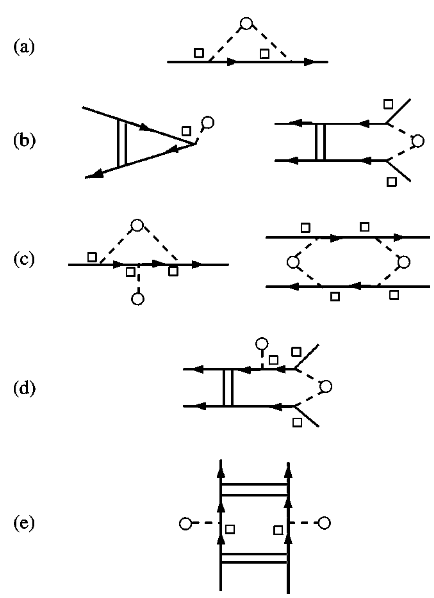

FIG. 15. One-loop diagrammatic contributions to the F terms. The diagrams in (a)–(e) represent contributions to F of order F, ΔF, F^2, $F^2\Delta$, and $F^2\Delta^2$, respectively.

n are summed. The pure random friction model contains only those diagrammatic contributions with no pinning disorder $\Delta=0$. Thus only the second and third terms in Eq. (E5) are taken into account for instance by the mapping to a polymer problem in the previous appendix. Note that there are no terms of $O(F\Delta^2)$ contributing to the renormalization of F, which follows diagrammatically because one needs at least two F's to put boxes (time derivatives) on the incoming legs originating from the two pinning disorder vertices.

A beautiful simplicity arises due to the extremely broad distribution of time scales in the RG, and can be seen from the structure of Eq. (E5). In particular, we note that the total number of powers of ω is conserved by all terms. Moreover, from the prior analysis, we expect $\eta^{(m)} \sim \eta_l^{2m^2-m}$. We thus conjecture that all $F_m^{(n)}$ scale in this way *independently* of n, i.e., $F_m^{(n)} \sim \eta_l^{2m^2-m}$ (see below to see that this scaling is indeed self-consistent). Under this assumption, the terms involving more than two $F's$ in Eq. (E5) can be seen to be strongly subdominant, which follows from the convexity of the $2m^2 - m$ factor in the exponential. Thus the superexponential (Gaussian) growth of the moments of the time scales, which is directly connected to the broad distribution of relaxation times, plays a key role in simplifying the structure of the RG.

One can thus restrict the analysis to the linear part in F of the full one-loop RG equation. Note that this indeed combines the effects of pinning disorder and the "upward" feedback of the random friction model—the first two terms in Eq. (E5). The linearized RG equation reads

$$\partial_l F^{(n)}_{p_1\cdots p_n} = \Gamma_l(2n^2-n)F^{(n)}_{p_1\cdots p_n} + (n+1)\alpha e^{-(d-2)l} \times \sum_{r=1}^{n}\sum_{q=1}^{p_r-1} F^{(n+1)}_{p_1\cdots p_{r-1}(p_r-q)p_{r+1}\cdots p_n q}, \tag{E6}$$

where $\alpha=\Lambda^{d-2}A_d$. The general study of this equation is again highly difficult but we see that it does have special solutions where the $F^{(n)}_{\{pi\}}$ depend only upon $m=\Sigma_i pi$. From the above consideration about the asymptotic behavior it is rather natural to look for such solutions.

Thus we let $F^{(n)}_{\{p_i\}}=F^{(n)}_{l,m}$. Then the linear RG equation (E6) becomes

$$\partial_l F^{(n)}_{l,m} = \Gamma_l(2n^2-n)F^{(n)}_{l,m} + (n+1)\alpha e^{-(d-2)l}(m-n)F^{(n+1)}_{l,m}. \tag{E7}$$

This is much easier to solve and the attentive reader will easily find that this infinite hierarchy of differential flow equations is solved asymptotically by the ansatz

$$F^{(n)}_{p_1\cdots p_n} = \left[\frac{\tilde{\beta}}{\alpha}e^{(d-2+\theta)l}\right]^{n-m}\left(\frac{\eta_l}{\eta_0}\right)^{2m^2-m}\tilde{\eta}_0^{(m)}(-1)^{m+1}a_m^n, \tag{E8}$$

where $m=\Sigma_{i=1}^n pi$. The a_m^n coefficients are given by

$$a_m^n = \frac{m!}{n!}\prod_{r=n}^{m-1}\frac{1}{2m+2r-1} = \frac{m\,!\,(2m+2n-3)\,!\,!}{n\,!\,(4m-3)\,!\,!}. \tag{E9}$$

The $\tilde{\eta}_0^{(m)}$ coefficients are not determined by the asymptotic analysis. In principle, they should be matched at some scale $l^*>0$ to the form of the $F_m^{(n)}$ coefficients determined by the early stages of renormalization, in which the linearized RG equation used to obtain them is not valid. One might imagine beginning with a model in which the bare relaxation time was distributed with cumulants $\eta_0^{(m)}$, and naively $\tilde{\eta}_0^{(m)} \approx \overline{\eta_0^m}$. We will use this prescription below purely in order to simplify notation. However, a more detailed analysis of the early stages of renormalization is in fact required to discern whether this is indeed correct.

2. Equation for the equilibrium response function

We are now in a position to extract the response function. Applying Eqs. (E4), (E8), (E9) for $n=1$ gives the response as an infinite series in $i\omega$:

$$R_k^{-1}(i\omega) = \sum_m (-1)^{m+1}\eta_0^{(m)}\exp\{(2m^2-m)U_k + [\gamma-(d-2+\theta)l - \ln(1/i\omega)]m\}, \tag{E10}$$

where $U_k \equiv U_{l=\ln(\Lambda/k)}$. Since it is not easy to resum this series, and its convergence properties are unclear, we now reformulate the above calculation in a functional way, in hopes of surmounting the limitations of the expansion in terms of moments of relaxation times. Let us introduce the generating function

$$G(i\omega,z) = \sum_{m\geq n\geq 1}(i\omega)^{m-n}z^n F^n_{ml}, \tag{E11}$$

which conveniently captures both the F term (and hence distribution of relaxation times) and the response function

$$F(z) = G(0,z), \tag{E12}$$

$$i\eta_0\omega + \Sigma_k(i\omega) = \frac{\partial}{\partial z}G(i\omega,z)\big|_{z=0}. \tag{E13}$$

Multiplying Eq. (E7) by the appropriate powers of z and $i\omega$ and summing gives the flow equation

$$\partial_l G = \Gamma_l(z\partial_z G + 2z^2\partial_z^2 G) + \alpha e^{-(d-2)l}\{i\omega\partial_{i\omega}[i\omega(\partial_z G - \partial_z G|_{z=0})]. \tag{E14}$$

This is somewhat simplified by defining the derivative $H(i\omega,z)=\partial_z G(i\omega,z)$,

$$\partial_l H = \Gamma_l(H + 5z\partial_z H + 2z^2\partial_z^2 H) + \alpha e^{-(d-2)l}i\omega\partial_{i\omega}(i\omega\partial_z H). \tag{E15}$$

A hopefully illuminating change of variables is to define

$$u = \ln(1/i\omega), \tag{E16}$$

$$v = \ln(z/i\omega) - (\theta+d-2)l \pm \ln(\alpha/\chi). \tag{E17}$$

Then $K(u,v)=H(i\omega,z)$, and obeys

$$\partial_{U_l}K = K + 3\partial_v K + 2\partial_v^2 K + (\partial_u - \partial_v)(e^{-v}\partial_v K), \quad \text{(E18)}$$

where $\partial_l U_l = \chi e^{\theta l}$ as before. Note that the F term is recovered in the limit $v \to \infty$ and $u - v = \ln z + \text{const}$ is fixed, in which the second term is negligible. In that limit, we recover the diffusion with drift equation (51), and the appropriate solution is $K(u,v) = \Phi(v-u) = F'(e^{v-u})$. More general solutions of Eq. (E18) remain to be found.

APPENDIX F: ONE LOOP HIERARCHY: METHOD OF CALCULATION

In this appendix we show how the systematic calculation of the one loop correction to the dynamical effective action can be organized, and sketch explicit calculation on the simplest examples. We focus on $T=0$.

The schematic form of the dynamical action $\mathcal{S}$ is given in the text in Eq. (77) as a sum of terms containing an increasing number of independent times (cumulants):

$$S = i\hat{u}_1\mathcal{S}_1 - \frac{1}{2}i\hat{u}_1 i\hat{u}_2\mathcal{S}_{12} - \frac{1}{6}i\hat{u}_1 i\hat{u}_2 i\hat{u}_3\mathcal{S}_{123} - \cdots. \quad \text{(F1)}$$

We use the same schematic notation where the indices $1,2,3,\ldots$, are short hand notations for $t_1,t_2,t_3,\ldots$, space coordinate and all time and space integrations are implicit. From Eq. (77) $\mathcal{S}_1$ is parametrized by an infinite set of kinetic coefficients $\overline{\eta}, D, \ldots, \mathcal{S}_{12}$ by a set of second cumulant functions $\Delta, G, A, B, C, \ldots, \mathcal{S}_{123}$ by a set of third cumulants $H, W, \ldots$, etc. [from Eq. (79)].

In a first stage we write the total one loop corrections to the action as the sum of tadpoles, two vertex loop, three vertex triangles, etc., with either $\mathcal{S}_{12}$ or $\mathcal{S}_{123}$ (and so on) type vertices using the full response function R_{12}, inverse of $i\hat{u}_1\mathcal{S}_1$, to contract the vertices (internal lines). Enumerating possible contractions and performing some combinatorics, yields upon grouping resulting terms by number of independent times:

$$\delta\mathcal{S}_1 = -\langle i\hat{u}_2\mathcal{S}_{12}\rangle, \quad \text{(F2)}$$

$$\delta\mathcal{S}_{12} = \langle i\hat{u}_3\mathcal{S}_{123}\rangle + \left[\frac{1}{2}\mathcal{S}_{34}\langle\mathcal{S}_{12}i\hat{u}_3 i\hat{u}_4\rangle + \langle\mathcal{S}_{24}i\hat{u}_3\rangle\langle\mathcal{S}_{13}i\hat{u}_4\rangle\right], \quad \text{(F3)}$$

$$\delta\mathcal{S}_{123} = \left[\frac{1}{2}\mathcal{S}_{45}\langle\mathcal{S}_{123}i\hat{u}_4 i\hat{u}_5\rangle + 3\langle\mathcal{S}_{14}i\hat{u}_5\rangle\langle\mathcal{S}_{235}i\hat{u}_4\rangle + \frac{3}{2}\langle\mathcal{S}_{12}i\hat{u}_4 i\hat{u}_5\rangle\mathcal{S}_{345}\right] + [2\langle\mathcal{S}_{34}i\hat{u}_5\rangle\langle\mathcal{S}_{15}i\hat{u}_6\rangle\langle\mathcal{S}_{26}i\hat{u}_4\rangle + 3\langle\mathcal{S}_{12}i\hat{u}_4 i\hat{u}_6\rangle\langle\mathcal{S}_{34}i\hat{u}_5\rangle\mathcal{S}_{56}], \quad \text{(F4)}$$

where additional (time) indices are integrated over Note that $\mathcal{S}_1$ is corrected only by tadpoles, $\mathcal{S}_{12}$ by tadpoles and two vertex loops, and so on.

In this formula notations such as, e.g., $\langle\mathcal{S}_{12}\hat{u}_3\hat{u}_4\rangle$ denote the sum of all possible contractions of the $\hat{u}$ fields with the u fields inside the bracket (at $T=0$ these are the only possible contractions). Since $\mathcal{S}_{12} = \mathcal{S}_{12}[u_{12}, \dot{u}_1, \dot{u}_2, \ddot{u}_1, \ddot{u}_2, \ldots]$ is an explicit function of $u_{12} = u_1 - u_2$, and time derivatives of u_1 and u_2 (and similarly for all other vertices) one may write the sum of all possible contractions as

$$\langle i\hat{u}_2\mathcal{S}_{12}\rangle = \left[R_{12}\partial_{u_{12}} + (\partial_1 R_{12})\partial_{\dot{u}_1} + (\partial_1^2 R_{12})\partial_{\ddot{u}_1}\right]\mathcal{S}_{12} + \cdots. \quad \text{(F5)}$$

We recall that here causality $R_{22}=0$ restricts contractions only with u_1 (first term), $\dot{u}_1$ (second term), etc.

The next stage is to make apparent $\overline{\eta}$, D, etc., and thus to define the expansion

$$R_{12} = \frac{1}{k^2}\left(\delta_{12} + \sum_{p=1}^{+\infty} A_p \partial_1^p \delta_{12}\right), \quad \text{(F6)}$$

$$k^2 R(s) = \frac{k^2}{k^2 + \Sigma(s)} = 1 + \sum_{n=1}^{+\infty}(-1)^n k^{-2n}\Sigma(s)^n = 1 + \sum_{p=1}^{+\infty} A_p s^p, \quad \text{(F7)}$$

$$A_1 = -k^{-2}\overline{\eta}, \; A_2 = -k^{-2}D + k^{-4}\overline{\eta}^2. \quad \text{(F8)}$$

The momentum structure of the one loop diagrams being trivial, within Wilson one can replace $k = \Lambda_l$ everywhere. As explained in the text this expansion in power of frequency can be done consistently and corresponds diagrammatically to expansion in number of dots.

One then evaluate the contractions shifting time integrations. Let us illustrate this on simple examples.

The corrections to η and D can be obtained from Eq. (F2) using Eq. (F5). One has

$$\delta\mathcal{S}_1 = -\left[R_{12}\partial_{u_{12}} + (\partial_1 R_{12})\partial_{\dot{u}_1} + (\partial_1^2 R_{12})\partial_{\ddot{u}_1}\right]\mathcal{S}_{12}, \quad \text{(F9)}$$

$$= -[(\delta_{12} + A_1\partial_1\delta_{12} + A_2\partial_1^2\delta_{12})\partial_{u_{12}}\mathcal{S}_{12} + (\partial_1\delta_{12} + A_1\partial_1^2\delta_{12})\partial_{\dot{u}_1}\mathcal{S}_{12} + \partial_1^2\delta_{12}\partial_{\ddot{u}_1}\mathcal{S}_{12}], \quad \text{(F10)}$$

$$= -\delta_{12}[(1 + A_1\partial_2 + A_2\partial_2^2)\partial_{u_{12}}\mathcal{S}_{12} + (\partial_2 + A_1\partial_2^2) \times \partial_{\dot{u}_1}\mathcal{S}_{12} + \partial_2^2\partial_{\ddot{u}_1}\mathcal{S}_{12}]. \quad \text{(F11)}$$

In the second line we have used the expansion (F6) and in the last line we have used that $\partial_1 R_{12} = -\partial_2 R_{12}$ and integrated by parts over t_2. Then time derivatives on the vertex $\mathcal{S}_{12}$ can be evaluated and tranformed into derivatives with respect to fields as

$$\partial_2 \equiv \partial_2\mathcal{S}_{12} = (-\dot{u}_2\partial_{u_{12}} + \ddot{u}_2\partial_{\dot{u}_2})\mathcal{S}_{12}, \quad \text{(F12)}$$

$$\partial_2^2 \equiv \partial_2^2\mathcal{S}_{12} = (-\ddot{u}_2\partial_{u_{12}} + \dot{u}_2^2\partial_{u_{12}}^2)\mathcal{S}_{12}. \quad \text{(F13)}$$

At all stages of the calculation we can drop all terms containing more than a fixed number (here 2) of time derivatives since they will contribute only to higher order terms in the effective action. Putting everything together we obtain

$$\delta\eta = G'(0) - \overline{\eta}\Delta''(0), \quad \text{(F14)}$$

$$\delta D = -A(0) + C'(0) - 2\overline{\eta}k^{-2}G'(0) - \Delta''(0)(k^{-2}D - k^{-4}\overline{\eta^2}), \tag{F15}$$

which, in terms of rescaled quantities, yield the Eqs. (90), (96) in the text.

Next we want to evaluate the corrections to second cumulant functions Δ, G, A, B, C encoded in $\delta\mathcal{S}_{12}$. We start by the simplest, the tadpole, which yields the feedback of third cumulants into second ones:

$$\delta|_{\text{tadpole}}S_{12} = \langle S_{123}\hat{u}_3\rangle = (R_{13}\partial_{u_1}S_{123} + \partial_1 R_{13}\partial_{\dot{u}_1}S_{123} + R_{23}\partial_{u_2}S_{123} + \partial_2 R_{23}\partial_{\dot{u}_2}S_{123}), \tag{F16}$$

$$= 2\delta_{13}[(1 + A_1\partial_3)\partial_{u_1} + \partial_3\partial_{\dot{u}_1}]S_{123}, \tag{F17}$$

$$= 2\delta_{13}[\partial_{u_1} + A_1\dot{u}_3\partial_{u_3}\partial_{u_1} + \dot{u}_3\partial_{u_3}\partial_{\dot{u}_1}]S_{123}, \tag{F18}$$

with $\partial_3 = \dot{u}_3\partial_{u_3}$. This gives the $2S_1'(0,0,u)$ term in Eq. (68) for Δ and the H feeding term in the equation for G (we have not explicitly computed the feeding of W into A, B, C but it is easily obtained from the above).

Next we need the $\mathcal{S}_{12}^2$ corrections to $\mathcal{S}_{12}$ which yield all non linear terms in Eqs. (91), (97)–(99) for G, A, B, C. The corresponding correction to $\delta\mathcal{S}_{12}$ consists in the two terms in the square bracket in Eq. (F3). The full calculation being tedious we only indicate here how one shuffles time integrals in the first term (denoted $\delta_1\mathcal{S}_{12}$). Starting from

$$\delta_1 S_{12} = \tfrac{1}{2}S_{34}\langle S_{12}\hat{u}_3\hat{u}_4\rangle = \tfrac{1}{2}S_{34}[(R_{14} - R_{24})\partial_{u_{12}} + \partial_1 R_{14}\partial_{\dot{u}_1} + \partial_2 R_{24}\partial_{\dot{u}_2} + \partial_1^2 R_{14}\partial_{\ddot{u}_1} + \partial_2^2 R_{24}\partial_{\ddot{u}_2}][(R_{13} - R_{23})\partial_{u_{12}} + \partial_1 R_{13}\partial_{\dot{u}_1} + \partial_2 R_{23}\partial_{\dot{u}_2} + \partial_1^2 R_{13}\partial_{\ddot{u}_1} + \partial_2^2 R_{23}\partial_{\ddot{u}_2}]S_{12}, \tag{F19}$$

using again identities such that $\partial_1 R_{14} = -\partial_4 R_{14}$, expanding the R and integrating by parts over t_4 and t_3 one can write

$$\delta_1 S_{12} = \tfrac{1}{2}S_{34}[(1 + A_1\overleftarrow{\partial}_4 + A_2\overleftarrow{\partial}_4^2)(\delta_{14} - \delta_{24})\partial_{u_{12}} + (\overleftarrow{\partial}_4 + A_1\overleftarrow{\partial}_4^2) \times(\delta_{14}\partial_{\dot{u}_1} + \delta_{24}\partial_{\dot{u}_2}) + \overleftarrow{\partial}_4^2(\delta_{14}\partial_{\ddot{u}_1} + \delta_{24}\partial_{\ddot{u}_2})] \times[(1 + A_1\overleftarrow{\partial}_3 + A_2\overleftarrow{\partial}_3^2)(\delta_{13} - \delta_{23})\partial_{u_{12}} + (\overleftarrow{\partial}_3 + A_1\overleftarrow{\partial}_3^2) \times(\delta_{13}\partial_{\dot{u}_1} + \delta_{23}\partial_{\dot{u}_2}) + \overleftarrow{\partial}_3^2(\delta_{13}\partial_{\ddot{u}_1} + \delta_{23}\partial_{\ddot{u}_2})]S_{12}. \tag{F20}$$

This is then in the form where, as above, all time derivatives can be replaced by derivatives over fields acting either on $\mathcal{S}_{12}$ or $\mathcal{S}_{34}$ using identities such as Eq. (F12) together with $\partial_4\partial_3\mathcal{S}_{34} = -\dot{u}_3\dot{u}_4\partial_{u_{34}}^2\mathcal{S}_{34}$. The evaluation of the second term in Eq. (F3) proceeds similarly and the sum of the two yields Eqs. (91), (97)–(99) in the text. Note that causality must be enforced at each step of the calculation.

APPENDIX G: INTEGRABLE "UNIRELAXATIONAL MODEL"

In this section we introduce a set of integrable models in various dimensions which can be used as a check of the FRG equations derived in this paper. This models have a remarkable property that despite being random the dynamics is extremely simple and the relaxation time scales are simply that of a single mode generalized oscillator.

Let us consider first the toy model in zero dimension

$$\eta(u_t)\partial_t u_t = f(u_t) - m^2 u_t, \tag{G1}$$

with $f(u) = -V'(u)$, when

$$\eta(u) = \overline{\eta}[1 - m^{-2}f'(u)], \tag{G2}$$

it can be rewritten

$$\partial_t[f(u_t) - m^2 u_t] = -\frac{m^2}{\overline{\eta}}[f(u_t) - m^2 u_t], \tag{G3}$$

which can be integrated exactly, yielding a pure exponential relaxation with a single time scale:

$$f(u_t) - m^2 u_t = e^{-t(m^2/\overline{\eta})}[f(u_0) - m^2 u_0]. \tag{G4}$$

Drawing $F(u) = f(u) - m^2 u$ as a function of u we see that all initial conditions starting in an interval between two adjacent minima and maxima and which contains a zero of $F(u)$, will converge exponentially to this zero of $F(u)$ [48]. Thus, if $F(u)$ has several zeroes the convergence will be to minima and maxima of the potential energy $H(u) = V(u) + 1/2\, m^2 u^2$ (depending on the initial condition). This should not be a surprise since in that case $\eta(u)$ changes sign so the dynamics is no more dissipative. One can then extend the system to nonzero temperature $T > 0$, imposing FDT with a stationary measure $e^{-H(u)/T}$ by adding ζ_t to the right-hand side of Eq. (G1) with correlations

$$\langle\zeta_t\zeta_{t'}\rangle = 2T\eta(u). \tag{G5}$$

Note that this means imaginary noise at points where $\eta(u)$ is negative. Thus the convergence to the maxima of $V(u)$ is killed by interference effects.

The exact response function associated to the u field can be obtained for this model at $T=0$. Upon adding an infinitesimal perturbation h_t on the right-hand side of Eq. (G1) the change δu is such that

$$[f'(u_t) - m^2]\delta u_t = -m^2\int_{t'} R_{tt'}^{(0)} h_{t'}, \tag{G6}$$

where u_t is given by Eq. (G4) and $R_{tt'}^{(0)} = (1/\overline{\eta})e^{-t(m^2/\overline{\eta})}$. The disorder averaged response function is thus

$$R_{tt'} = \overline{\left(1 - \frac{f'(u_t)}{m^2}\right)^{-1}} R_{tt'}^{(0)}. \tag{G7}$$

In the large time limit time translational invariance is restored and one finds

$$R_{tt'} = R^{(0)}_{tt'}. \tag{G8}$$

This a consequence of the following property:

$$\overline{\left(1 - \frac{f'(u_t)}{m^2}\right)^{-1}} = 1, \tag{G9}$$

where for each realization of the random function $f(u)$, u_t is the solution of

$$f(u_t) = m^2 u_t, \tag{G10}$$

and the average is taken with respect to any translationally invariant distribution for $f(u)$.

A similar model may be introduced in arbitrary dimension. It is defined as

$$\overline{\eta}\dot{F}_{rt} = -F_{rt} + \zeta_{rt}, \tag{G11}$$

$$F_{rt} = \nabla^2 u_{rt} + f(u_{rt}, r). \tag{G12}$$

This yields the equation of motion

$$-\overline{\eta}\nabla^2 \dot{u}_{rt} - \overline{\eta} f'(u_{rt}, r)\dot{u}_{rt} = \nabla^2 u_{rt} + f(u_{rt}, r) + \zeta_{rt}. \tag{G13}$$

This is identical in form to the model discussed in the text with the exception that the damping coefficient is wave vector q dependent and vanishes as q^2. Upon averaging over disorder one obtains an MSR action identical to the model studied in the text apart from the q^2 mean damping with

$$G(u) = \overline{\eta}\Delta'(u), \tag{G14}$$

$$A(u) = -\overline{\eta}^2\Delta''(u), \tag{G15}$$

and no other higher order vertex for a Gaussian distributed f (more general expressions can be easily obtained for non-Gaussian distributions). The bare response function of this model factors as

$$R_{q\omega} = \frac{1}{q^2(1 + \overline{\eta} i\omega)}. \tag{G16}$$

Similar arguments as above yield that this is also the exact response.

The one loop Wilson FRG of this model is very similar to the one performed for the model in the text. Since the relaxation time is dimensionless in this model vertices such as G and A scale identically to Δ. Hence the appropriate rescaled functions for these vertices are $\tilde{G} \sim \Lambda_l^{-\epsilon} G$, $\tilde{A} \sim \Lambda_l^{-\epsilon} A$. The one loop FRG equations are identical to the one given in the text for $\tilde{G}$ and $\tilde{A}$ apart from the (linear) rescaling part (not involving ζ) being identical to that for $\tilde{\Delta}$. One can then check that the relation

$$\tilde{G}(u) = \tilde{\Delta}'(u), \tag{G17}$$

$$\tilde{A}(u) = -\overline{\eta}2\tilde{\Delta}''(u), \tag{G18}$$

specific to this model, are indeed exactly preserved by the FRG, as announced in the text. Computing the correction to the self-energy yields

$$\partial_l \Sigma(\omega) = A_d \Lambda_l^{d-2}\{\Delta''(0)[R(\omega) - R(0)] + G'(0)[2i\omega R(\omega) - i\omega R(0)] + A(0)\omega^2 R(\omega)\}. \tag{G19}$$

One then checks that this exactly vanishes using $A(0) = -\overline{\eta}^2\Delta''(0)$, $G'(0) = \overline{\eta}\Delta''(0)$ and the above exact form for $R(\omega)$.

This model can be further generalized to include second time derivative terms $D \neq 0$. Adding the term $D\ddot{F}_{rt}$ to the left-hand side of Eq. (G11) one obtains the model in the text with $\overline{\eta} \to q^2\overline{\eta}$ and $D \to q^2 D$ in the bare inverse response function. Similar arguments yield invariance of the FRG function within the manifold (100) given in the text (third cumulants have also been included).

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[35] Except for the massless $q=0$ mode, which is not important.

[36] The idea is that by inversion symmetry of the averaged action spatial gradient corrections must start as k^2 multiplied by some power of ω and can thus always be neglected compared to the elastic term k^2.

[37] In that case $D>\overline{\eta}^2/4$ corresponds to underdamped dynamics and no determined sign for the response function [consistent with the connected moment (58) becoming negative for $D>\overline{\eta}^2/2$].

[38] Perturbation theory (e.g., within the Wilson mode integration) can be equivalently performed, splitting as usual $S=S_{\text{quad}}+S_{\text{int}}$, either (i) including no kinetic term in the quadratic part of the action S_{quad} (using then $R^{\text{quad}}_{ktt'}=\delta_{tt'}/q^2$ but considering in all internal response lines the chain of graphs represented in Fig. 12, with multiple $n=1$, $m\geqslant 1$ F vertices on internal lines), (ii) including just $\overline{\eta}$ in the quadratic part [using then $R^{\text{quad}}_k(i\omega)=1/(q^2+i\overline{\eta}\omega)$ and considering all graphs with multiple $n=1$, $m\geqslant 2$ F vertices on internal lines], or (iii) considering the total quadratic part [e.g., using the full $R_{k,l}(i\omega)$ and only $n\geqslant 2$ vertices].

[39] G. Schehr and P. Le Doussal, cond-mat/0304486; P. Chauve *et al.*, Phys. Rev. E **64**, 051102 (2001).

[40] Note that for depinning the G term in Eq. (91) yields $G\sim\eta\Delta^2$ and gives a part of the two loop $O(\epsilon^2)$ correction to z of the same order than the one coming from the direct correction $\eta\Delta^2$ to η to two loops.

[41] Consider the equation of motion $1/2\,\partial_u D(u_{rt},r)\dot{u}^2_{rt}+D(u_{rt},r)\ddot{u}_{rt}=\nabla^2 u_{rt}+f(u_{rt},r)$. Multipying by $\dot{u}_{rt}$ we note that it has conserved "total energy" $\partial_t E=0$ with $E=\int_r\big[H[u_{rt},r]+1/2\,D(u_{rt},r)\dot{u}^2_{rt}\big]$. It thus corresponds to some Hamiltonian classical dynamics (with a nonlinear mass term). The associated MSR dynamical action obeys the continuous global λ symmetry. This, however, is not FDT, even if we choose $\lambda=1/T$, meaning that if we add the standard terms (11) it would not satisfy FDT. The reason for that can be traced to the fact that the boundary term obtained upon applying the symmetry (112) is $[E(t_f)-E(t_i)]/T$ and thus contain extra time derivatives which invalidate the arguments made in this section. It is possible, however, to add additional noise at the boundary, i.e., a bulk term $-T\int_{rt}\partial_t[D(u_{rt},r)(i\hat{u}_{rt})^2]$, so that FDT is satisfied, i.e., the extra time derivative term is cancelled upon integration over $\hat{u}_{rt}$ at the boundary. This is relevant also for the disorder case, since averaging over disorder one gets, an $m=2$ term with C a first derivative, $B=C'/2$ and $A=0$. This indeed is the condition for invariance of the $m=2$ term under the continuous global λ symmetry.

[42] Another symmetry often considered associated to FDT is $i\hat{u}_{rt}\to -i\hat{u}_{r,-t}+(T\eta)^{-1}(\delta H/\delta u)|_{u_{r,-t}}$, $u_{rt}\to u_{r,-t}$. This is a symmetry of the unaveraged dynamical effective action. It is useful, e.g., to study [50] consequences of the accidental FDT property of the one-dimensional KPZ equation. It seems however a priori less useful in order to study the effective action once one integrates over modes, i.e., in the context of Wilson FRG (also further complications arise after averaging over disorder). It may be worth studying within the exact RG context, which is beyond the scope of the present paper.

[43] As a curiosity note the exact (implicit) solution for $\theta=2$ obtained by writing Eq. (A6) as $\widetilde{\beta}d(k^{-2})/d\Sigma=2k^{-2}+2(i\omega+\Sigma)^{-1}$, which yields $\Sigma_k(i\omega)=-(\widetilde{\beta}/2)\ln[k^2(1+2\int_0^{\Sigma}d\lambda e^{-2\lambda/\widetilde{\beta}}(i\omega+\lambda)^{-1}]$.

[44] Note that since $dy/dg=(1+g)e^g$ formally this integral reads simply $\int dg e^{g+\mathcal{Y}ge^g}$ but the contour remains to be worked out.

[45] Ideally one should also take into account the influence of modes with wave vectors $q<k$ on the relaxation of mode k since these are obviously left out of the RG scheme. However, simple arguments suggest that this influence decays fast: the influence of the jumps in the mode q on the mode k may decay as fast as $e^{-k/q}$. At least this is necessary condition for such droplet arguments to make sense.

[46] P. Le Doussal, C. Monthus, and D. S. Fisher, Phys. Rev. E **59**, 4795 (1999); L. Laloux and P. Le Doussal, *ibid.* **57**, 6296 (1998).

[47] P. P. Mitra and P. Le Doussal, Phys. Rev. B **44**, 12 035 (1991).

[48] Note, however, that if the initial condition starts in an interval two adjacent minima and maxima which does not contains a zero of $F(u)$, it instead converges in finite time to one of the endpoint of the interval at which $\eta(u)$ vanishes. The solution in Eq. (G4) analytically continues the relaxation of the force to zero.

[49] P. Le Doussal and C. Monthus, Physica A **317**, 140 (2003).

[50] E. Frey, U. Tauber, and T. Hwa, cond-mat/9601049.

[51] B. Altshuler *et al.*, JETP Lett. **45**, 199 (1987); Zh. Eksp. Teor. Fiz. **94**, 258 (1988).